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On some solutions to generalized spheroidal wave equations and applications

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Abstract

Expansions in series of Coulomb and hypergeometric functions for the solutions of the generalized spheroidal wave equations (GSWEs) are analysed and written together in pairs. Each pair consists of a solution in series of hypergeometric functions and another in series of Coulomb wavefunctions and has the same recurrence relations for the series coefficients, but the solutions inside it present different radii of convergence. Expansions without a phase parameter are derived by truncating the series with a phase parameter. For the Whittaker–Hill equation, solutions are found by treating that equation as a particular case of GSWE while, for the confluent GSWE, solutions, with and without a phase parameter, are given as pairs of series of Coulomb wavefunctions. Amongst the applications there are equations for the time dependence of Dirac test fields in some nonflat Friedmann–Robertson–Walker spacetimes, the radial Schrödinger equation for an electron in the field of two Coulombian centres and the Schrödinger equation for the Razavy-type quasi-exactly solvable potentials. For these problems it is possible to find wavefunctions in terms of infinite series, regular and convergent over the entire range of the independent variable, by matching expansions belonging to one or more of the above pairs. The infinite-series solutions for the Razavy-type potentials, in addition to the polynomial ones, suggest that the whole energy spectra may be determined without appealing to perturbation theory or semi-classical methods of approximation.

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1. Generalities

In this paper we deal with solutions to the generalized spheroidal wave equations (GSWEs) and their particular cases. We also discuss some possible applications of the results found here. Before outlining what we are doing, we present some ideas concerning GSWEs which are used throughout the paper.

For definiteness, we adopt the Leaver version

$$x(x-x_0)\frac{\mathrm{d}^2U}{\mathrm{d}x^2}+(B_1+B_2x)\frac{\mathrm{d}U}{\mathrm{d}x}+[B_3-2\omega\eta(x-x_0)+\omega^2x(x-x_0)]U=0, \tag{1}$$

for the GSWE [1], where x_0, B_i, η and ω are constants. If $\eta=0$ and $x_0\neq 0$, then we have the ordinary spheroidal wave equation. On the other hand, supposing that

$$B_1=-x_0/2, \qquad B_2=1, \qquad x=x_0\cos^2(u) \tag{2}$$

in equation (1), we find

$$\frac{\mathrm{d}^2U}{\mathrm{d}u^2}+[-4B_3-4\eta\omega x_0+4\eta\omega x_0\cos(2u)+\omega^2x_0^2\sin^2(2u)]U=0, \tag{3}$$

which is the Whittaker–Hill equation (WHE) [2]. Since the WHE has just three parameters, we may absorb x_0 into ω. A third particular case, the confluent GSWE, occurs when $x_0=0$

$$x^2\frac{\mathrm{d}^2U}{\mathrm{d}x^2}+(B_1+B_2x)\frac{\mathrm{d}U}{\mathrm{d}x}+[B_3-2\omega\eta x+\omega^2x^2]U=0, \tag{4}$$

with both the singular points $x=0$ and $x=\infty$ being irregular [1].

As usual, we consider only solutions given as series of special functions with three-term recurrence relations for the series coefficients. If there are no free constants in the GSWE, the series convergence demands the presence of a phase parameter ν which must be determined from a characteristic equation ensuing from the recurrence relations. Series expansions with a phase parameter are double-sided with the summation index n running from $-\infty$ to ∞. However, the GSWEs may also admit solutions in finite series. For the WHE these solutions are known as Ince's polynomials [3], whereas for the general case they can be called Heun's polynomials, since the GSWE is a confluent Heun equation and the confluent GSWE is a double confluent Heun equation [4]. Furthermore, from a known solution $S(x)$ with a phase parameter ν

$$S(x):=U(B_1,B_2,B_3;\nu,x_0,\omega,\eta;x), \tag{5}$$

(where ':=' means 'equal by definition') it may be possible to obtain new solutions by means of one or more of the following transformation rules [4,5]—T_1, T_2, T_3—

$$T_1S(x)=x^{1+B_1/x_0}U(C_1,C_2,C_3;\nu_1,x_0,\omega,\eta;x), \tag{6a}$$

$$T_2S(x)=(x-x_0)^{1-B_2-B_1/x_0}U(B_1,D_2,D_3;\nu_2,x_0,\omega,\eta;x), \tag{6b}$$

where

$$C_1:=-B_1-2x_0, \qquad C_2:=2+B_2+\frac{2B_1}{x_0},$$
$$C_3:=B_3+\left(1+\frac{B_1}{x_0}\right)\left(B_2+\frac{B_1}{x_0}\right), \tag{7a}$$

$$D_2:=2-B_2-\frac{2B_1}{x_0}, \qquad D_3:=B_3+\frac{B_1}{x_0}\left(\frac{B_1}{x_0}+B_2-1\right). \tag{7b}$$

These rules are valid only for $x_0\neq 0$ and they can be demonstrated by setting

$$U=x^{1+B_1/x_0}f_1, \qquad U=(x-x_0)^{1-B_2-B_1/x_0}f_2$$

into equation (1). They must be applied to general solutions of the GSWE in which no values are specified for the parameters; it would make no sense to try to apply them to a solution of the WHE, for instance. A further rule, now valid also for $x_0=0$, is

$$T_3S(x)=U(B_1,B_2,B_3;\nu_3,x_0,-\omega,-\eta;x), \qquad \forall x_0, \tag{8}$$

in which it is assumed that we have to change the sign of (η, ω) only where these quantities appear explicitly, preserving the expressions for the other constants. In effect, the solutions regarded here will have the forms $U = e^{i\omega x} g$ and $e^{-i\omega x} h$ and thereupon we get

$$x(x - x_0)\frac{d^2 g}{dx^2} + [B_1 + B_2 x + 2i\omega x(x - x_0)]\frac{dg}{dx} + [B_3 + i\omega B_1 + i\omega B_2 x - 2\omega\eta(x - x_0)]g = 0,$$

$$x(x - x_0)\frac{d^2 h}{dx^2} + [B_1 + B_2 x - 2i\omega x(x - x_0)]\frac{dh}{dx} + [B_3 - i\omega B_1 - i\omega B_2 x - 2\omega\eta(x - x_0)]h = 0,$$

for g and h with the sole changes stated above. If we do not take into account this remark, we would get wrong results for the solutions of the Teukolsky equations, for example, where the constants depend on η and ω (see, for example, [1]). With this proviso, the rule T_3 will not be used explicitly and it is put here just to remind that for each written solution, another solution exists. Moreover, these rules in general also transform the phase parameter, although that will not happen for the solutions discussed here.

With regard to the confluent GSWE, for which T_1 and T_2 do not work, we have the rules t_1 and t_2 [1,4]

$$t_1 S(x) = e^{i\omega x + B_1/(2x)} x^{-i\eta - B_2/2} U(B'_1, B'_2, B'_3; \omega', \eta'; \vartheta), \tag{9a}$$

$$t_2 S(x) = e^{B_1/x} x^{2-B_2} U(\overline{B}_1, \overline{B}_2, \overline{B}_3; \omega, \eta; x), \tag{9b}$$

where

$$B'_1 = \omega B_1, \qquad B'_2 = 2 + 2i\eta, \qquad B'_3 = B_3 - \left(\frac{B_2}{2} + i\eta\right)\left(\frac{B_2}{2} - i\eta - 1\right),$$
$$\omega' = 1, \qquad i\eta' = \frac{B_2}{2} - 1, \qquad \vartheta = \frac{iB_1}{2x}, \tag{10a}$$

and

$$\overline{B}_1 = -B_1, \qquad \overline{B}_2 = 4 - B_2, \qquad \overline{B}_3 = B_3 + 2 - B_2. \tag{10b}$$

An additional procedure, which is used to obtain solutions without phase parameters out of those with phase parameters, consists in truncating the series with the phase parameter, that is, restricting the summation index n to non-negative values. In this process ν will become determined regardless of the characteristic equation and, consequently, the truncation is allowed only if there is some arbitrary constant in the differential equation. In general we obtain more than one expression for ν. Besides this, once we have obtained one solution without a phase parameter, new ones can be generated from the transformation rules.

All the facts exposed above are well known in the theory of Heun's differential equations of which the GSWEs are particular cases, as mentioned before. We use these to obtain explicit solutions to the GSWEs in series of Gauss hypergeometric and Coulomb wavefunctions. We do not give just one solution of type (5) but also the solutions arising from it via the transformations rules. This procedure requires some more space but it is necessary if we want to use the solutions to solve particular equations. On the other hand, we pay special attention to the solution truncation for, in general, this process leads to more than one (three in our case) possible form to the recurrence relations for the series coefficients.

Firstly, in section 2, we deal with the solutions with phase parameters. The expansions in hypergeometric functions are taken from [6] with minor modifications; the series in Coulomb wavefunctions are the Leaver solutions [1] and those which come from them by rule T_2. The solutions are written as two pairs, each pair exhibiting the same series coefficient and containing an expansion in hypergeometric functions and another in Coulomb functions. For the WHE one pair is even with respect to the variable u and the other is odd. The idea of working simultaneously with these two types of expansions appears in Otchik [7], who proposed to match them in order to solve the radial Teukolsky equations (see also [8–11]). Therefore, this section can be seen as a transposition of Otchik's approach to other problems described by non-confluent GSWEs. Actually, we find that our results may be used to obtain solutions to the time dependence of massive-Dirac test fields in radiation-dominated Friedmann–Robertson–Walker (FRW) spacetimes.

In section 3.1, the solutions found in section 2 are truncated. This provides three values for ν in each pair of solutions. We select two of them and keep four pairs without phase parameters. As an application we examine the solutions of the radial Schrödinger equation for an electron in the field of two Coulombian centres (the two-centre problem) and conclude that it is possible to construct solutions which are regular over the entire range of the radial coordinate by matching expansions in hypergeometric functions with expansions in Coulomb wavefunctions. This procedure offers the advantages of not presenting a phase parameter to be interpreted, and of operating with one-sided series. A new solution to the angular equation is also found. In section 3.2 we regard the case in which $B_2 = 1$ and $B_1/x_0 = -1/2$ (here called Whittaker–Hill-type) and find that for the WHE, properly, the expansions in hypergeometric functions coincide with the four Arscott expansions in trigonometric functions [2] but, now, for each of them we have a partner in series of Coulomb functions. This fact enables us to match solutions of a given pair to obtain the complete energy spectrum for the Schrödinger equation with quasi-exactly solvable (QES) Razavy-type potentials without the need of perturbation theory or semi-classical methods of approximations.

In section 4 the Leaver solutions in series of Coulomb functions to the confluent GSWE are duplicated by the rule t_2. We find that such expansions may be used to obtain solutions for the time dependence of massive-Dirac test fields in dust-dominated FRW spacetimes. The truncated expansions are applied to the Schrödinger equation with asymmetric double-Morse potentials. For QES potentials we obtain polynomial solutions. In section 5, there are concluding remarks and the appendix shows us how to obtain the the recurrence relations for the truncated solutions.

2. Solutions with phase parameters

By U_1^ν and U_2^ν we denote the two expansions in series of hypergeometric functions and by $\tilde{U}_1^\nu$ and $\tilde{U}_2^\nu$ the two expansions in series of Coulomb wavefunctions. The superscript ν indicates that they depend on a phase parameter ν. By demanding invariance of solutions under the operations implied by rules T_1 and T_2, we get $\tilde{U}_2^\nu$ as a new expansion resulting from the Leaver one, $\tilde{U}_1^\nu$. However, by requiring that the series coefficients for U_1^ν and U_2^ν are identical to those which appear in $\tilde{U}_1^\nu$ and $\tilde{U}_2^\nu$, we are compelled to redefine the phase parameters of the original expansions in hypergeometric functions. This gives the two pairs of solutions (U_1^ν, $\tilde{U}_1^\nu$) and (U_2^ν, $\tilde{U}_2^\nu$), each with the same series coefficients. We first give the general solutions, then we restrict these to Whittaker–Hill-type equations and finally discuss the Dirac equation for radiation-dominated FRW backgrounds.

2.1. General case

The expansions in series of Coulomb wavefunctions are written explicitly as series of the regular (or Kummer) and irregular (or Tricomi) confluent hypergeometric functions $M(a, b; z)$ and

$U(a, b, z)$, respectively, rather than in terms of regular and irregular Coulomb wavefunctions $F_{n+\nu}$ and $G_{n+\nu}$. As a matter of fact we use $\tilde{M}(a, b; z)$

$$\tilde{M}(a, b; z) := \frac{\Gamma(b-a)}{\Gamma(b)} M(a, b; z) = \frac{\Gamma(b-a)}{\Gamma(b)} \left(1 + \frac{a}{b} z + \frac{a(a+1)}{2!b(b+1)} z^2 + \cdots\right) \tag{11a}$$

instead of $M(a, b; z)$. If, for brevity, we define $\mathcal{F}(a, b; z)$ as

$$\mathcal{F}(a_n, b_n; z) := U(a_n, b_n; z) \text{ or } (-1)^n \tilde{M}(a_n, b_n; z), \tag{11b}$$

the first pair of solutions assumes the form

$$\begin{aligned} U_1^\nu &= e^{i\omega x} \sum_{n=-\infty}^{\infty} b_n F\left(\frac{B_2}{2} - n - \nu - 1, n + \nu + \frac{B_2}{2}; B_2 + \frac{B_1}{x_0}; \frac{x_0 - x}{x_0}\right), \\ \tilde{U}_1^\nu &= e^{i\omega x} x^{\nu+1-(B_2/2)} \sum_{n=-\infty}^{\infty} b_n (-2i\omega x)^n \mathcal{F}(n + \nu + 1 + i\eta, 2n + 2\nu + 2; -2i\omega x), \end{aligned} \tag{12a}$$

with the following recurrence relations for the coefficients b_n

$$\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \tag{12b}$$

where

$$\begin{aligned} \alpha_n &= i\omega x_0 \frac{(n + \nu + 2 - (B_2/2))(n + \nu + 1 - (B_2/2) - (B_1/x_0))(n + \nu + 1 - i\eta)}{2(n + \nu + 1)(n + \nu + 3/2)}, \\ \beta_n &= -B_3 - \eta\omega x_0 - \left(n + \nu + 1 - \frac{B_2}{2}\right)\left(n + \nu + \frac{B_2}{2}\right) \\ &\quad - \frac{\eta\omega x_0((B_2/2) - 1)((B_2/2) + (B_1/x_0))}{(n + \nu)(n + \nu + 1)}, \\ \gamma_n &= -i\omega x_0 \frac{(n + \nu + (B_2/2) - 1)(n + \nu + (B_2/2) + (B_1/x_0))(n + \nu + i\eta)}{2(n + \nu - 1/2)(n + \nu)}. \end{aligned} \tag{12c}$$

The phase parameter ν may be determined from a characteristic equation given as a sum of two infinite continued fractions, namely

$$\beta_0 = \frac{\alpha_{-1}\gamma_0}{\beta_{-1}-} \frac{\alpha_{-2}\gamma_{-1}}{\beta_{-2}-} \frac{\alpha_{-3}\gamma_{-2}}{\beta_{-3}-} + \cdots + \frac{\alpha_0\gamma_1}{\beta_1-} \frac{\alpha_1\gamma_2}{\beta_2-} \frac{\alpha_2\gamma_3}{\beta_3-} \cdots. \tag{12d}$$

Using the rule T_2 we obtain the second pair of solutions

$$\begin{aligned} U_2^\nu &= f \sum_{n=-\infty}^{\infty} b'_n F\left(-n - \nu - \frac{B_2}{2} - \frac{B_1}{x_0}, n + \nu + 1 - \frac{B_2}{2} - \frac{B_1}{x_0}; 2 - B_2 - \frac{B_1}{x_0}; \frac{x_0 - x}{x_0}\right), \\ \tilde{U}_2^\nu &= f x^{\nu+(B_2/2)+(B_1/x_0)} \sum_{n=-\infty}^{\infty} b'_n (-2i\omega x)^n \, \mathcal{F}(n + \nu + 1 + i\eta, 2n + 2\nu + 2; -2i\omega x), \end{aligned} \tag{13a}$$

where

$$f := e^{i\omega x}(x - x_0)^{1-B_2-(B_1/x_0)}, \tag{13b}$$

and

$$\begin{aligned} \alpha'_n &= i\omega x_0 \frac{(n + \nu + 1 + (B_2/2) + (B_1/x_0))(n + \nu + (B_2/2))(n + \nu + 1 - i\eta)}{2(n + \nu + 1)(n + \nu + 3/2)}, \\ \beta'_n &= \beta_n, \\ \gamma'_n &= -i\omega x_0 \frac{(n + \nu - (B_2/2) - (B_1/x_0))(n + \nu + 1 - (B_2/2))(n + \nu + i\eta)}{2(n + \nu - 1/2)(n + \nu)}, \end{aligned} \tag{13c}$$

in the recurrence relations

$$\alpha'_n b'_{n+1} + \beta'_n b'_n + \gamma'_n b'_{n-1} = 0.$$

The characteristic equation is again given by equation (12*d*) because we have $\beta'_n = \beta_n$ and $\alpha'_n \gamma'_{n+1} = \alpha_n \gamma_{n+1}$. Moreover, by applying the rule T_1 to $(U_2^\nu, \tilde{U}_2^\nu)$ we return to $(U_1^\nu, \tilde{U}_1^\nu)$, and thus both sets of solutions are closed under applications of T_1 and T_2. We also remark that the above forms for the expansions in Gauss hypergeometric were obtained by accomplishing the replacements

$$\nu \to \nu + 1 - \frac{B_2}{2}, \qquad \nu' \to \nu + \frac{B_2}{2} + \frac{B_1}{x_0},$$

into the original solutions of [6]. These substitutions have permitted us to see that both solutions depend on the same phase parameter ν which, in turn, is the very one that appears in the expansions $\tilde{U}_1^\nu$ and $\tilde{U}_2^\nu$.

The series in terms of Coulomb wavefunctions are convergent for $|x| > |x_0|$ [1] while those in terms of hypergeometric functions do not converge at $|x| = \infty$. In effect, following the steps sketched in [6] or [10] we find

$$\lim_{n\to\infty} \frac{b_{n+1}F_{n+1}}{b_n F_n} = \lim_{n\to-\infty} \frac{b_n F_n}{b_{n+1}F_{n+1}} = \frac{i\omega x_0}{2|n|}\left[\frac{2x}{x_0} - 1 + \sqrt{\frac{4}{x_0^2}x(x-x_0)}\right]$$

where $F_n := F((B_2/2) - n - \nu - 1, n + \nu + (B_2/2), B_2 + (B_1/x_0); y)$. Therefore, the ratio test implies that the expansion U_1^ν converges in any finite region of the complex plane. This is the same for U_2^ν. Note that in the case of polynomial solutions the ratio test becomes meaningless.

It is worth mentioning that the $n \geqslant 0$ part of the expansions in regular confluent hypergeometric functions is convergent for all values of x [1]. Furthermore, there are some properties of confluent hypergeometric functions regarding only these functions that are useful here. Firstly, while $M(a, b; 0) = 1$, in general $U(a, b; z)$ has a logarithmical behaviour when $z \to 0$ [16] and this will make the expansions in irregular confluent hypergeometric functions inadequate for obtaining polynomial solutions. Secondly, as $|z| \to \infty$ we have [12]

$$M(a, b; z) = \begin{cases} \dfrac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b}[1 + O(|z|^{-1})] & (\Re z > 0), \\ \dfrac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}[1 + O(|z|^{-1})] & (\Re z < 0), \end{cases} \tag{14}$$

and we must take these properties into account when we examine the asymptotic behaviour of solutions. Moreover, if a is a negative integer, $\tilde{M}(a, b; z)$ is a polynomial, suggesting that the expansions in series of regular hypergeometric functions are suitable to obtain polynomial solutions (i.e., in finite series) as seen in sections 3.2.1 and 4.2.2.

2.2. Limits for Whittaker–Hill-type equations

For $B_2 = 1$, $B_1 = -x_0/2$, we define c_n by means of $b'_n = 2(n + \nu + 1/2)c_n$ and find that the recurrence relations for b_n and c_n become identical. Therefore, we may set $c_n = b_n$ and then the solutions acquire the forms

$$\begin{aligned} U_1^\nu &= e^{i\omega x} \sum_{n=-\infty}^{\infty} b_n F\left(-n - \nu - \frac{1}{2}, n + \nu + \frac{1}{2}; \frac{1}{2}; \frac{x_0 - x}{x_0}\right), \\ \tilde{U}_1^\nu &= e^{i\omega x} x^{\nu+1/2} \sum_{n=-\infty}^{\infty} b_n (-2i\omega x)^n \mathcal{F}(n + \nu + 1 + i\eta, 2n + 2\nu + 2; -2i\omega x), \end{aligned} \tag{15}$$

$$U_2^\nu = e^{i\omega x}(x-x_0)^{1/2}\sum_{n=-\infty}^{\infty}\left(n+\nu+\frac{1}{2}\right)b_n F\left(-n-\nu, n+\nu+1; \frac{3}{2}; \frac{x_0-x}{x_0}\right),$$
$$\tilde{U}_2^\nu = e^{i\omega x}(x-x_0)^{1/2}x^\nu\sum_{n=-\infty}^{\infty}(n+\nu+\tfrac{1}{2})b_n(-2i\omega x)^n\mathcal{F}(n+\nu+1+i\eta, 2n+2\nu+2; -2i\omega x), \tag{16}$$

with the following simplified coefficients in the recurrence relations

$$\alpha_n = \frac{i\omega x_0}{2}(n+\nu+1-i\eta),$$
$$\beta_n = -B_3 - \eta\omega x_0 - (n+\nu+\tfrac{1}{2})^2, \tag{17}$$
$$\gamma_n = -\frac{i\omega x_0}{2}(n+\nu+i\eta).$$

For a WHE we have $x = x_0\cos^2 u$, $(x_0-x)/x_0 = \sin^2(u)$ and the hypergeometric functions in U_1^ν and U_2^ν can be written as trigonometric functions by means of [12]

$$F(-a, a; 1/2; \sin^2 u) = \cos(2au), \qquad F(a, 1-a; 3/2; \sin^2 u) = \frac{\sin[(2a-1)u]}{(2a-1)\sin(u)}. \tag{18}$$

Thus, except for a multiplicative constant, the solutions of the WHE are given by

$$U_1^\nu = e^{(i/2)\omega x_0\cos(2u)}\sum_{n=-\infty}^{\infty} b_n\cos[(2n+2\nu+1)u],$$
$$\tilde{U}_1^\nu = e^{(i/2)\omega x_0\cos(2u)}(\cos u)^{2\nu+1}\sum_{n=-\infty}^{\infty} b_n(-2i\omega x_0\cos^2 u)^n \tag{19}$$
$$\times\,\mathcal{F}(n+\nu+1+i\eta, 2n+2\nu+2; -2i\omega x_0\cos^2 u),$$

$$U_2^\nu = e^{(i/2)\omega x_0\cos(2u)}\sum_{n=-\infty}^{\infty} b_n\sin[(2n+2\nu+1)u],$$
$$\tilde{U}_2^\nu = e^{(i/2)\omega x_0\cos(2u)}(\cos u)^{2\nu}\sin u\sum_{n=-\infty}^{\infty}(n+\nu+\tfrac{1}{2})b_n(-2i\omega x_0\cos^2 u)^n \tag{20}$$
$$\times\,\mathcal{F}(n+\nu+1+i\eta, 2n+2\nu+2; -2i\omega x_0\cos^2 u),$$

where the first pair is constituted by even solutions and the second pair by odd solutions.

2.2.1. Dirac equation in radiation-dominated FRW spacetimes. As an illustration we consider the Dirac equation ($\hbar = c = 1$) for test fields with mass μ in nonflat FRW spacetimes, since the equations for the time dependence have no free parameters. The line element in its conformally static form is

$$ds^2 = [A(\tau)]^2\left[d\tau^2 - d\chi^2 - \frac{\sin^2(\sqrt{\epsilon}\chi)}{\epsilon}(d\theta^2 + \sin^2\theta\, d\varphi^2)\right], \tag{21}$$

where $\epsilon = \pm 1$ is the spatial curvature. If the Dirac spinor Ψ is redefined as

$$\Omega(\tau,\chi,\theta,\phi) := A^{3/2}\sin(\sqrt{\epsilon}\chi)\sqrt{\sin\theta}\;\Psi(\tau,\chi,\theta,\phi), \tag{22}$$

its time dependence is given by [13]

$$i\frac{dP(\tau)}{d\tau} = \sigma P(\tau) - \mu A(\tau)Q(\tau),$$
$$i\frac{dQ(\tau)}{d\tau} = -\sigma Q(\tau) - \mu A(\tau)P(\tau), \tag{23}$$

where P and Q are two spinor components and σ is a separation constant. For $\epsilon = 1$, σ is any half-integer different from $\pm 1/2$ [14] and, for $\epsilon = -1$, σ is any nonvanishing real number. On the other hand, taking $S = Q - P$ and $T = P + Q$, the preceding equations yield

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau} + \mathrm{i}\mu A(\tau)\right] S(\tau) = \mathrm{i}\sigma T(\tau),$$
$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau} - \mathrm{i}\mu A(\tau)\right] T(\tau) = \mathrm{i}\sigma S(\tau), \tag{24}$$

which implies

$$\frac{\mathrm{d}^2 S}{\mathrm{d}\tau^2} + \left[\sigma^2 + \mathrm{i}\mu \frac{\mathrm{d}A(\tau)}{\mathrm{d}\tau} + \mu^2 A^2(\tau)\right] S = 0, \tag{25}$$

$$T = \frac{1}{\mathrm{i}\sigma}\left[\frac{\mathrm{d}}{\mathrm{d}\tau} + \mathrm{i}\mu A(\tau)\right] S. \tag{26}$$

For radiation-dominated models the scale factor is given by $A(\tau) = a_0 \sin(\sqrt{\epsilon}\tau)/\sqrt{\epsilon}$ and so equation (25) assumes the form

$$\frac{\mathrm{d}^2 S}{\mathrm{d}\tau^2} + [\sigma^2 + \mathrm{i}\mu a_0 \cos(\sqrt{\epsilon}\tau) + \epsilon\mu^2 a_0^2 \sin^2(\sqrt{\epsilon}\tau)] S = 0. \tag{27}$$

This is a WHE with $2u = \sqrt{\epsilon}\tau$ and the transformation $x = \cos^2(\sqrt{\epsilon}\tau/2)$ brings it to Leaver's form for the GSWE

$$x(x-1)\frac{\mathrm{d}^2 S}{\mathrm{d}x^2} + \left(x - \frac{1}{2}\right)\frac{\mathrm{d}S}{\mathrm{d}x} + \left[-\epsilon(\sigma^2 + \mathrm{i}\mu a_0) + 4\mu^2 a_0^2 x(x-1) - 2\mathrm{i}\mu a_0 \epsilon(x-1)\right] S = 0.$$

Thus, the parameters appearing in equation (1) can be written as

$$x_0 = 1, \qquad B_1 = -1/2, \qquad B_2 = 1,$$
$$B_3 = -\epsilon(\sigma^2 + \mathrm{i}\mu a_0), \qquad \omega = \pm 2\mu a_0, \qquad \mathrm{i}\eta = \mp\epsilon/2.$$

If $\epsilon = 1$, we have $0 \leqslant x \leqslant 1$ and the solutions must be written in series of trigonometric functions which are regular and convergent in this interval. The full wavefunctions Ψ will diverge at the spacetime singular point $\tau = 0$, but this is due to the factor $A^{-(3/2)}$ in equation (22). For $\epsilon = -1$, we have $1 \leqslant x < \infty$ and the solutions may be formed by matching both solutions in each pair (with $\mathcal{F} = U$), since at the singular point $x = 1$ only the series in hyperbolic functions converge while for $x \to \infty$ only the expansions in Coulomb wavefunctions converge. The divergence of Ψ at $x = 1$ results again from the factor $A^{-(3/2)}(\tau)$ and not from divergence in the solutions to the WHE. Note moreover that both signs for (η, ω) are allowed and thus we may obtain four solutions as required if we want to have a complete basis for the solutions of Dirac equation (the spatial equations afford only one solution for a given set of quantum numbers).

There is also a nonsingular spacetime with $\epsilon = -1$ where we have $B_2 = 1$ and $B_1 = -x_0/2$ but not a WHE. Hence we could use the solutions given by equations (15)–(17). This spacetime can also be interpreted as a radiation-dominated FRW model with a negative effective pressure. Its scale factor is $A(\tau) = a_0 \cosh\tau$ [15] and therefore

$$\frac{\mathrm{d}^2 S}{\mathrm{d}\tau^2} + [\sigma^2 + \mathrm{i}\mu a_0 \sinh\tau + \mu^2 a_0^2 \cosh^2\tau] S = 0. \tag{28}$$

This is not a WHE because the sinh and the cosh have interchanged positions and the equation is not symmetric under $\tau \leftrightarrow -\tau$. Writing $t = a_0 \sinh\tau$ for the coordinate time $\mathrm{d}t = A(\tau)\,\mathrm{d}\tau$ and performing the change of variable $x = t + \mathrm{i}a_0$ we get the GSWE

$$x(x - 2\mathrm{i}a_0)\frac{\mathrm{d}^2 S}{\mathrm{d}x^2} + (x - \mathrm{i}a_0)\frac{\mathrm{d}S}{\mathrm{d}x} + [\sigma^2 - \mu a_0 + \mathrm{i}\mu(x - 2\mathrm{i}a_0) + \mu^2 x(x - 2\mathrm{i}a_0)] S = 0, \tag{29a}$$

and hence

$$x_0 = 2\mathrm{i}a_0, \qquad B_1 = -\mathrm{i}a_0, \qquad B_2 = 1,$$
$$B_3 = \sigma^2 - \mu a_0, \qquad \omega = \pm\mu, \qquad \eta = \mp\mathrm{i}/2. \tag{29b}$$

Again we have to match solutions, since for $\sinh^2\tau \leqslant 1$ ($\Leftrightarrow |x| \leqslant |x_0|$) only the expansions in series of hypergeometric functions converge, whereas for $\tau \to \infty$ only the expansions in Coulomb wavefunctions converge.

3. Solutions without phase parameters

Supposing that there is some free parameter in the GSWE, we truncate the solutions with phase parameters, that is, we take $n \geqslant 0$. Firstly, we present the solutions for the general case and their possible applications to the angular and radial equations of the two-centre problem. Then we restrict the results for the case $B_2 = 1$, $B_1 = -x_0/2$ and show how these solutions can be applied to find the wavefunction for the Schrödinger equation with QES Razavy-type potentials.

3.1. General case

The solutions obtained from the truncation of the expansions given in section 2.1 are displayed in four pairs denoted by $(U_i, \tilde{U}_i)$, $i = 1, 2, 3, 4$. Starting from one pair, the others can be derived by means of the rules T_1 and T_2 according to the scheme

$$(U_1, \tilde{U}_1) \overset{T_1}{\longleftrightarrow} (U_2, \tilde{U}_2) \overset{T_2}{\longleftrightarrow} (U_3, \tilde{U}_3) \overset{T_1}{\longleftrightarrow} (U_4, \tilde{U}_4) \overset{T_2}{\longleftrightarrow} (U_1, \tilde{U}_1) \tag{30a}$$

which corresponds to

$$\nu_1 = \frac{B_2}{2} - 1 \overset{T_1}{\longleftrightarrow} \nu_2 = \frac{B_1}{x_0} + \frac{B_2}{2} \overset{T_2}{\longleftrightarrow} \nu_3 = 1 - \frac{B_2}{2} \overset{T_1}{\longleftrightarrow} \nu_4 = -\frac{B_1}{x_0} - \frac{B_2}{2} \overset{T_2}{\longleftrightarrow} \nu_1. \tag{30b}$$

Note that there are solutions with opposite signs for ν; therefore, if in one pair a denominator of the recurrence relations is zero (integer or half-integer value for ν), in another pair the denominator is well defined. The recurrence relations and the characteristic equations (for solutions in infinite series) have one of the three forms given below. The first case ($\alpha_{-1} = 0$) is the general one and the others ($\alpha_{-1} \neq 0$) may occur only for special values for the parameters.

$$\left.\begin{aligned} &\alpha_0 b_1 + \beta_0 b_0 = 0, \\ &\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \quad (n \geqslant 1), \end{aligned}\right\} \Rightarrow \beta_0 = \frac{\alpha_0\gamma_1}{\beta_1 -}\,\frac{\alpha_1\gamma_2}{\beta_2 -}\,\frac{\alpha_2\gamma_3}{\beta_3 -}\cdots \tag{31}$$

$$\left.\begin{aligned} &\alpha_0 b_1 + \beta_0 b_0 = 0, \\ &\alpha_1 b_2 + \beta_1 b_1 + \left[\alpha_{-1} + \gamma_1\right] b_0 = 0, \\ &\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \quad (n \geqslant 2), \end{aligned}\right\} \Rightarrow \beta_0 = \frac{\alpha_0\left[\alpha_{-1} + \gamma_1\right]}{\beta_1 -}\,\frac{\alpha_1\gamma_2}{\beta_2 -}\,\frac{\alpha_2\gamma_3}{\beta_3 -}\cdots \tag{32}$$

$$\left.\begin{aligned} &\alpha_0 b_1 + \left[\beta_0 + \alpha_{-1}\right] b_0 = 0, \\ &\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \quad (n \geqslant 1), \end{aligned}\right\} \Rightarrow \beta_0 + \alpha_{-1} = \frac{\alpha_0\gamma_1}{\beta_1 -}\,\frac{\alpha_1\gamma_2}{\beta_2 -}\,\frac{\alpha_2\gamma_3}{\beta_3 -}\cdots. \tag{33}$$

In each pair the hypergeometric functions can be rewritten as Jacobi's polynomials $P_n^{(\alpha,\beta)}(z)$ by using the formula [16]

$$F(-n, n+1+\alpha+\beta; 1+\alpha; y) = \frac{n!}{(1+\alpha)_n} P_n^{(\alpha,\beta)}(1-2y), \tag{34a}$$

where $(1+\alpha)_n$ denotes the Pocchammer symbol defined as

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \qquad (a)_0 = 1. \tag{34b}$$

Therefore, the truncated expansions in hypergeometric functions are solutions of the Fackerell–Crossman type [17]. In fact, the solutions U_1^ν and U_2^ν were obtained in [6] as generalizations of a Fackerell–Crossman solution which now is recovered together with other solutions. We first write the four pairs of solutions, relegating their derivations to appendix, and then discuss some applications. Note that for the truncated solutions we have $n \geqslant -1$ in α_n, $n \geqslant 0$ in β_n and $n \geqslant 1$ in γ_n.

First pair: $\nu = (B_2/2) - 1$ *in* $(U_1^\nu, \tilde{U}_1^\nu)$.

$$\begin{aligned} U_1 &= \mathrm{e}^{\mathrm{i}\omega x}\sum_{n=0}^{\infty} b_n^{(1)} F\left(-n, n+B_2-1; B_2+\frac{B_1}{x_0}; \frac{x_0-x}{x_0}\right), \\ \tilde{U}_1 &= \mathrm{e}^{\mathrm{i}\omega x}\sum_{n=0}^{\infty} b_n^{(1)} (-2\mathrm{i}\omega x)^n \mathcal{F}\left(n+\frac{B_2}{2}+\mathrm{i}\eta, 2n+B_2; -2\mathrm{i}\omega x\right), \end{aligned} \tag{35a}$$

$$\begin{aligned} \alpha_n^{(1)} &= \mathrm{i}\omega x_0 \frac{(n+1)(n-(B_1/x_0))(n+(B_2/2)-\mathrm{i}\eta)}{2(n+(B_2/2))(n+(B_2/2)+1/2)}, \\ \beta_n^{(1)} &= -B_3 - \eta\omega x_0 - n(n+B_2-1) - \frac{\eta\omega x_0((B_2/2)-1)((B_2/2)+(B_1/x_0))}{(n+(B_2/2)-1)(n+B_2/2)}, \\ \gamma_n^{(1)} &= -\mathrm{i}\omega x_0 \frac{(n+B_2-2)(n+B_2+(B_1/x_0)-1)(n+(B_2/2)-1+\mathrm{i}\eta)}{2(n+(B_2/2)-3/2)(n+(B_2/2)-1)}. \end{aligned} \tag{35b}$$

Recurrence relations: if $B_2 = 1$, equation (32); if $B_2 = 2$, equation (33); otherwise, equation (31).

Second pair: $\nu = (B_2/2) + (B_1/x_0)$ *in* $(U_1^\nu, \tilde{U}_1^\nu)$ *or* $(U_1, \tilde{U}_1) \xrightarrow{T_1} (U_2, \tilde{U}_2)$.

$$\begin{aligned} U_2 &= \mathrm{e}^{\mathrm{i}\omega x} x^{1+(B_1/x_0)} \sum_{n=0}^{\infty} b_n^{(2)} F\left(-n, n+1+B_2+\frac{2B_1}{x_0}; B_2+\frac{B_1}{x_0}; \frac{x_0-x}{x_0}\right), \\ \tilde{U}_2 &= \mathrm{e}^{\mathrm{i}\omega x} x^{1+(B_1/x_0)} \sum_{n=0}^{\infty} b_n^{(2)} (-2\mathrm{i}\omega x)^n \mathcal{F}\left(n+1+\mathrm{i}\eta+\frac{B_2}{2}+\frac{B_0}{x_0}, 2n+2+B_2+\frac{2B_1}{x_0}; -2\mathrm{i}\omega x\right), \end{aligned} \tag{36a}$$

$$\begin{aligned} \alpha_n^{(2)} &= \mathrm{i}\omega x_0 \frac{(n+1)(n+2+(B_1/x_0))(n+1+(B_2/2)+(B_1/x_0)-\mathrm{i}\eta)}{2(n+1+(B_2/2)+(B_1/x_0))(n+(3/2)+(B_2/2)+(B_1/x_0))}, \\ \beta_n^{(2)} &= -B_3 - \eta\omega x_0 - \left(n+1+\frac{B_1}{x_0}\right)\left(n+B_2+\frac{B_1}{x_0}\right) \\ &\quad - \frac{\eta\omega x_0((B_2/2)-1)((B_2/2)+(B_1/x_0))}{(n+(B_2/2)+(B_1/x_0))(n+1+(B_2/2)+(B_1/x_0))}, \\ \gamma_n^{(2)} &= -\mathrm{i}\omega x_0 \frac{(n+B_2+(B_1/x_0)-1)(n+B_2+(2B_1/x_0))(n+(B_2/2)+(B_1/x_0)+\mathrm{i}\eta)}{2(n-(1/2)+(B_2/2)+(B_1/x_0))(n+(B_2/2)+(B_1/x_0))}. \end{aligned} \tag{36b}$$

Recurrence relations: if $(B_2/2)+(B_1/x_0) = 0$, equation (33); if $(B_2/2)+(B_1/x_0) = -1/2$, equation (32); otherwise, equation (31).

Third pair: $\nu = 1 - (B_2/2)$ *in* $(U_2^{\nu}, \tilde{U}_2^{\nu})$ *or* $(U_2, \tilde{U}_2) \xrightarrow{T_2} (U_3, \tilde{U}_3)$.

$$U_3 = e^{i\omega x}(x - x_0)^{1-B_2-(B_1/x_0)} x^{1+(B_1/x_0)} \sum_{n=0}^{\infty} b_n^{(3)} F\left(-n, n+3-B_2; 2-B_2-\frac{B_1}{x_0}; \frac{x_0 - x}{x_0}\right),$$

$$\tilde{U}_3 = e^{i\omega x}(x - x_0)^{1-B_2-(B_1/x_0)} x^{1+(B_1/x_0)} \sum_{n=0}^{\infty} b_n^{(3)} (-2i\omega x)^n \times \mathcal{F}\left(n+2-\frac{B_2}{2}+i\eta, 2n+4-B_2; -2i\omega x\right), \tag{37a}$$

$$\alpha_n^{(3)} = i\omega x_0 \frac{(n+1)(n+2+(B_1/x_0))(n+2-(B_2/2)-i\eta)}{2(n+2-(B_2/2))(n+(5/2)-(B_2/2))},$$

$$\beta_n^{(3)} = -B_3 - \eta\omega x_0 - (n+1)(n+2-B_2) - \frac{\eta\omega x_0((B_2/2)-1)((B_2/2)+(B_1/x_0))}{(n+1-(B_2/2))(n+2-(B_2/2))}, \tag{37b}$$

$$\gamma_n^{(3)} = -i\omega x_0 \frac{(n+2-B_2)(n+1-B_2-(B_1/x_0))(n+1-(B_2/2)+i\eta)}{2(n+(1/2)-(B_2/2))(n+1-(B_2/2))}.$$

Recurrence relations: if $B_2 = 2$, equation (33); if $B_2 = 3$, equations (32); otherwise, equation (31).

Fourth pair: $\nu = -(B_2/2) - (B_1/x_0)$ *in* $(U_2^{\nu}, \tilde{U}_2^{\nu})$ *or* $(U_3, \tilde{U}_3) \xrightarrow{T_1} (U_4, \tilde{U}_4)$.

$$U_4 = e^{i\omega x}(x - x_0)^{1-B_2-(B_1/x_0)} \sum_{n=0}^{\infty} b_n^{(4)} F\left(-n, n+1-B_2-\frac{2B_1}{x_0}; 2-B_2-\frac{B_1}{x_0}; \frac{x_0 - x}{x_0}\right),$$

$$\tilde{U}_4 = e^{i\omega x}(x - x_0)^{1-B_2-(B_1/x_0)} \sum_{n=0}^{\infty} b_n^{(4)} (-2i\omega x)^n \times \mathcal{F}\left(n+1+i\eta-\frac{B_2}{2}-\frac{B_0}{x_0}, 2n+2-B_2-\frac{2B_1}{x_0}; -2i\omega x\right), \tag{38a}$$

$$\alpha_n^{(4)} = i\omega x_0 \frac{(n+1)(n-(B_1/x_0))(n+1-(B_2/2)-(B_1/x_0)-i\eta)}{2(n+1-(B_2/2)-(B_1/x_0))(n+(3/2)-(B_2/2)-(B_1/x_0))},$$

$$\beta_n^{(4)} = -B_3 - \eta\omega x_0 - \left(n - \frac{B_1}{x_0}\right)\left(n - B_2 + 1 - \frac{B_1}{x_0}\right) - \frac{\eta\omega x_0((B_2/2)-1)((B_2/2)+(B_1/x_0))}{(n-(B_2/2)-(B_1/x_0))(n+1-(B_2/2)-(B_1/x_0))}, \tag{38b}$$

$$\gamma_n^{(4)} = -i\omega x_0 \frac{(n+1-B_2-(B_1/x_0))(n-B_2-(2B_1/x_0))(n-(B_2/2)-(B_1/x_0)+i\eta)}{2(n-(1/2)-(B_2/2)-(B_1/x_0))(n-(B_2/2)-(B_1/x_0))}.$$

Recurrence relations: if $(B_2/2) + (B_1/x_0) = 0$, equation (33); if $(B_2/2) + (B_1/x_0) = 1/2$, equation (32); otherwise, equation (31).

Note that, in each pair, to get the expressions for $(\alpha_n, \beta_n, \gamma_n)$ the shortest way is to insert the value for ν into the nontruncated expressions. To obtain $(U_i, \tilde{U}_i)$ and the recurrence relations it is easier to use the transformations rules, since this leads the hypergeometric functions to be already in a polynomial form as above.

3.1.1. The angular and radial equations for the two-centre problem. Now we comment upon how the earlier solutions can be applied to the angular and radial equations of the two-centre problem. Our starting point and conventions are taken from Leaver [1]. The wavefunction ψ of the time-independent Schrödinger equation has the form

$$\psi = e^{im\varphi}\overline{R}(\lambda)\overline{S}(\mu), \qquad \lambda := (r_1 + r_2)/(2a), \qquad \mu := (r_1 - r_2)/(2a), \tag{39a}$$

where m is any integer, r_1 and r_2 are the distances from the electron to the two centres, and $2a$ is the intercentre distance. By performing the changes of variables

$$\begin{aligned} S(x) &= x^{m/2}(2-x)^{m/2}f^{-}(x), \qquad x = \mu + 1, (0 \leqslant x \leqslant 2), \\ R(x) &= x^{m/2}(x-2)^{m/2}f^{+}(x), \qquad x = \lambda + 1, (x \geqslant 2), \end{aligned} \tag{39b}$$

where $S(x) = \overline{S}(\lambda),\ R(x) = \overline{R}(\mu)$. Leaver obtained GSWEs for $f^{\pm}$ with

$$\begin{aligned} &x_0 = 2, \qquad \omega^2 = 2a^2E, \qquad \omega\eta^{\pm} = -a(N_1 \pm N_2), \qquad B_1 = -2(m+1), \\ &B_2 = 2(m+1), \qquad B_3^{\pm} = \omega^2 + 2a(N_1 \pm N_2) + m(m+1) - A_{lm}. \end{aligned} \tag{39c}$$

A_{lm} is a separation constant, whereas N_1 and N_2 are related to the values of the two charges. We are assuming that $N_1 \pm N_2 \neq 0$. To have regular wavefunctions when $m \geqslant 0$ we employ the solutions $(U_1, \tilde{U}_1)$ to the GSWE and thus

$$\begin{aligned} S_1 &= e^{i\omega x}x^{m/2}(2-x)^{m/2}\sum_{n=0}^{\infty} b_n^{-} F\left(-n, n+2m+1; m+1; 1-\frac{x}{2}\right), \\ \tilde{S}_1 &= e^{i\omega x}x^{m/2}(2-x)^{m/2}\sum_{n=0}^{\infty} b_n^{-}(2i\omega x)^n \tilde{M}(n+m+1+i\eta^{-}, 2n+2m+2; -2i\omega x), \end{aligned} \tag{40a}$$

and

$$\begin{aligned} R_1 &= e^{i\omega x}x^{m/2}(x-2)^{m/2}\sum_{n=0}^{\infty} b_n^{+} F\left(-n, n+2m+1; m+1; 1-\frac{x}{2}\right), \\ \tilde{R}_1 &= e^{i\omega x}x^{m/2}(x-2)^{m/2}\sum_{n=0}^{\infty} b_n^{+}(-2i\omega x)^n U(n+m+1+i\eta^{+}, 2n+2m+2; -2i\omega x), \end{aligned} \tag{40b}$$

where the recurrence relations for $b_n^{\pm}$ are given by equation (31) with

$$\begin{aligned} \alpha_n^{\pm} &= i\omega\frac{(n+1)(n+m+1-i\eta^{\pm})}{(n+m+3/2)}, \\ \beta_n^{\pm} &= \beta_n = A_{ml} - \omega^2 - m(m+1) - n(n+2m+1), \\ \gamma_n^{\pm} &= -i\omega\frac{(n+2m)(n+m+i\eta^{\pm})}{(n+m-1/2)}. \end{aligned} \tag{40c}$$

If we rewrite $S_1(x)$ in terms of associated Legendre polynomials, we recognize $S_1(x)$ as a Barber–Hassé solution [18] but now we also have a representation in series of regular Coulomb wavefunctions (constructed originally for a radial equation). The solution $\tilde{R}_1(x)$ for the radial equation is regular and convergent anywhere except at $x = 2$, the point at which the solution $R_1(x)$ is regular and convergent. Therefore, we can match them in order to get solutions for the radial wavefunction. This seems to be a possible alternative to the treatment of [19] which proposes matching expansions in Coulomb wavefunctions (with phase parameters) and Jaffé's expansions (without phase parameters), each of them having different characteristic equations. Furthermore, we can again express R_1 as series of associated Legendre polynomials. Then it becomes obvious that we are matching solutions of Barber–Hassé type (originally conceived for the angular equation) with solutions in series of Coulomb wavefunctions.

If $m \leqslant 0$, regular and convergent solutions may be formed from the pair $(U_3, \tilde{U}_3)$ and the sole difference consists in the change of m by $-m$ in equations (40a)–(40c). Therefore, it is sufficient to put $|m|$ where we had m in those solutions, but not in equation (39a). We could also use the pairs $(U_2, \tilde{U}_2)$ and $(U_4, \tilde{U}_4)$ and this would not modify the results. For example, the angular solutions constructed from $(U_2, \tilde{U}_2)$ have the form

$$S_2(x) = e^{i\omega x} x^{-(m/2)}(2-x)^{m/2} \sum_{n=m}^{\infty} b_n F\left(-n, n+1; m+1; 1-\frac{x}{2}\right),$$
$$\tilde{S}_2(x) = e^{i\omega x} x^{-(m/2)}(2-x)^{m/2} \sum_{n=m}^{\infty} b_n (2i\omega x)^n \tilde{M}(n+1+i\eta^-, 2n+2; -2i\omega x)$$

where in the recurrence relations for b_n, equation (31),

$$\alpha_n = i\omega \frac{(n+1-m)(n+1-i\eta^-)}{(n+3/2)},$$
$$\beta_n = \beta_n = A_{ml} - \omega^2 - m(m+1) - (n-m)(n+m+1),$$
$$\gamma_n = -i\omega \frac{(n+m)(n+i\eta^-)}{(n-1/2)}.$$

These solutions differ from the previous ones inasmuch as the sum begins at $n = m$ by reason of $\alpha_{m-1} = 0$. However, if we perform the substitution $n \to n+m$, use the relation $F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$ and rename the coefficients, we notice that these solutions are identical to $(S_1, \tilde{S}_1)$.

We have found two possible representations for the angular dependence of the two-centre problem. A similar fact occurs with the angular Teukolsky equations. In effect, the angular wavefunctions has the form

$$S(x) = x^{(1/2)|m-s|}(2-x)^{(1/2)|m+s|} f(x), \qquad 0 \leqslant x = 1 + \cos\theta \leqslant 2,$$

where $f(x)$ obeys a GSWE with $B_1 = -2|m-s| - 2$, $B_2 = |m+s| + |m-s| + 2$, $x_0 = 2$. The pair of solutions $(U_1, \tilde{U}_1)$ gives

$$f_1 = e^{i\omega x} \sum_{n=0}^{\infty} b_n^{(1)} F\left(-n, n+|m+s|+|m-s|+1; |m+s|+1; \frac{2-x}{2}\right),$$
$$\tilde{f}_1 = e^{i\omega x} \sum_{n=0}^{\infty} b_n^{(1)} (2i\omega x)^n \tilde{M}\left(n + \frac{|m+s|}{2} + \frac{|m-s|}{2} + 1 + i\eta, 2n + |m+s| + |m-s| + 2; -2i\omega x\right). \qquad (41a)$$

The first solution is one of the Fackerell–Crossman solutions of the angular Teukolsky equations and the second is a new representation in series of Coulomb wavefunctions. The second Fackerell–Crossman solution and its partner can be derived from the above solutions by the transformation rule T_3. Once more we may obtain identical solutions starting from $(U_2, \tilde{U}_2)$ but then, similarly to the two-centre problem, the sum will begin at $n = |m-s|$.

3.2. Limits for Whittaker–Hill-type equations

For this particular case, similar to the case with phase parameters, all the recurrence relations become simpler since there are no denominators in them. For the WHE the expansions in series of hypergeometric functions reduce again to series of trigonometric functions which are not but the Arscott solutions [2]. Note that now each pair presents a different form for the recurrence relations. The term $-\alpha_{-1}$ in equation (48*c*), instead of $+\alpha_{-1}$, comes from the redefinition of the series coefficients. The four pairs are written below.

First pair.

$$U_1 = e^{i\omega x}\sum_{n=0}^{\infty} b_n^{(1)} F\left(-n, n; \frac{1}{2}; \frac{x_0 - x}{x_0}\right),$$
$$\tilde{U}_1 = e^{i\omega x}\sum_{n=0}^{\infty} b_n^{(1)}(-2i\omega x)^n \mathcal{F}(n + \tfrac{1}{2} + i\eta, 2n + 1; -2i\omega x), \tag{42a}$$

where

$$\alpha_n^{(1)} = \frac{i\omega x_0}{2}\left(n + \frac{1}{2} - i\eta\right), \qquad \beta_n^{(1)} = -n^2 - B_3 - \eta\omega x_0,$$
$$\gamma_n^{(1)} = -\frac{i\omega x_0}{2}\left(n - \frac{1}{2} + i\eta\right), \tag{42b}$$

in the recurrence relations

$$\left.\begin{aligned} &\alpha_0 b_1 + \beta_0 b_0 = 0,\\ &\alpha_1 b_2 + \beta_1 b_1 + [\alpha_{-1} + \gamma_1]\, b_0 = 0,\\ &\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \quad (n \geqslant 2), \end{aligned}\right\} \Rightarrow \beta_0 = \frac{\alpha_0[\alpha_{-1} + \gamma_1]}{\beta_1 -}\,\frac{\alpha_1\gamma_2}{\beta_2 -}\,\frac{\alpha_2\gamma_3}{\beta_3 -}\cdots. \tag{42c}$$

For the WHE we have two even solutions:

$$U_1 = e^{(i\omega/2)\cos(2u)}\sum_{n=0}^{\infty} b_n^{(1)}\cos(2nu),$$
$$\tilde{U}_1 = e^{(i\omega/2)\cos(2u)}\sum_{n=0}^{\infty} b_n^{(1)}(-2i\omega x_0\cos^2 u)^n \mathcal{F}(n + \tfrac{1}{2} + i\eta, 2n + 1; -2i\omega x_0\cos^2 u). \tag{43}$$

Second pair.

$$U_2 = e^{i\omega x}x^{1/2}\sum_{n=0}^{\infty} b_n^{(2)} F\left(-n, n + 1; \frac{1}{2}; \frac{x_0 - x}{x_0}\right),$$
$$\tilde{U}_2 = e^{i\omega x}x^{1/2}\sum_{n=0}^{\infty} b_n^{(2)}(-2i\omega x)^n \mathcal{F}(n + i\eta + 1, 2n + 2; -2i\omega x), \tag{44a}$$

where

$$\frac{2\alpha_n^{(2)}}{i\omega x_0} = (n + 1 - i\eta), \qquad \beta_n^{(2)} = -(n + \tfrac{1}{2})^2 - B_3 - \eta\omega x_0, \qquad \frac{2\gamma_n^{(2)}}{i\omega x_0} = -(n + i\eta), \tag{44b}$$

in the recurrence relation

$$\left.\begin{aligned} &\alpha_0 b_1 + [\beta_0 + \alpha_{-1}]b_0 = 0,\\ &\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0\ (n \geqslant 1), \end{aligned}\right\} \Rightarrow \beta_0 + \alpha_{-1} = \frac{\alpha_0}{\beta_1 -}\,\frac{\alpha_1\gamma_2}{\beta_2 -}\,\frac{\alpha_2\gamma_3}{\beta_3 -}\cdots. \tag{44c}$$

Again the solutions to the WHE are even:

$$U_2 = e^{(i\omega/2)\cos(2u)}\sum_{n=0}^{\infty} b_n^{(2)}\cos[(2n + 1)u],$$
$$\tilde{U}_2 = e^{(i\omega/2)\cos(2u)}\cos u\sum_{n=0}^{\infty} b_n^{(2)}(-2i\omega x_0\cos^2 u)^n \mathcal{F}(n + 1 + i\eta, 2n + 2; -2i\omega x_0\cos^2 u). \tag{45}$$

Third pair.

$$U_3 = e^{i\omega x}(x-x_0)^{1/2}x^{1/2}\sum_{n=0}^{\infty}(n+1)b_n^{(3)}F\left(-n, n+2; \frac{3}{2}; \frac{x_0-x}{x_0}\right),$$
$$\tilde{U}_3 = e^{i\omega x}(x-x_0)^{1/2}x^{1/2}\sum_{n=0}^{\infty}(n+1)b_n^{(3)}(-2i\omega x)^n\mathcal{F}(n+\tfrac{3}{2}+i\eta, 2n+3; -2i\omega x), \tag{46a}$$

$$\frac{2\alpha_n^{(3)}}{i\omega x_0} = \left(n+\frac{3}{2}-i\eta\right), \qquad \beta_n^{(3)} = -(n+1)^2 - B_3 - \eta\omega x_0, \qquad \frac{2\gamma_n^{(3)}}{i\omega x_0} = -\left(n+\frac{1}{2}+i\eta\right). \tag{46b}$$

$$\left.\begin{array}{l}\alpha_0 b_1 + \beta_0 b_0 = 0,\\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geqslant 1),\end{array}\right\} \Rightarrow \beta_0 = \frac{\alpha_0\gamma_1}{\beta_1 -}\ \frac{\alpha_1\gamma_2}{\beta_2 -}\ \frac{\alpha_2\gamma_3}{\beta_3 -}\cdots. \tag{46c}$$

Now the solutions to the WHE are odd:

$$U_3 = e^{(i\omega/2)\cos(2u)}\sum_{n=0}^{\infty}b_n^{(3)}\sin[(2n+2)u],$$
$$\tilde{U}_3 = e^{(i\omega/2)\cos(2u)}\sin(2u)\sum_{n=0}^{\infty}(n+1)b_n^{(3)}(-2i\omega x_0\cos^2 u)^n \times \mathcal{F}(n+\tfrac{3}{2}+i\eta, 2n+3; -2i\omega x_0\cos^2 u). \tag{47}$$

Fourth pair.

$$U_4 = e^{i\omega x}(x-x_0)^{1/2}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})b_n^{(4)}F(-n, n+1; \tfrac{3}{2}; y),$$
$$\tilde{U}_4 = e^{i\omega x}(x-x_0)^{1/2}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})b_n^{(4)}(2i\omega x_0 y)^n\mathcal{F}(n+1+i\eta, 2n+2; -2i\omega x), \tag{48a}$$

with

$$\alpha_n^{(4)} = \alpha_n^{(2)}, \qquad \beta_n^{(4)} = \beta_n^{(2)}, \qquad \gamma_n^{(4)} = \gamma_n^{(2)}, \text{ see equation (44}b\text{)}, \tag{48b}$$

in the recurrence relations (note the minus sign before α_{-1})

$$\left.\begin{array}{l}\alpha_0 b_1 + [\beta_0 - \alpha_{-1}]b_0 = 0,\\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geqslant 1),\end{array}\right\} \Rightarrow \beta_0 - \alpha_{-1} = \frac{\alpha_0}{\beta_1 -}\ \frac{\alpha_1\gamma_2}{\beta_2 -}\ \frac{\alpha_2\gamma_3}{\beta_3 -}\cdots. \tag{48c}$$

Again the solutions to the WHE are odd:

$$U_4 = e^{(i\omega/2)\cos(2u)}\sum_{n=0}^{\infty}b_n^{(4)}\sin[(2n+1)u],$$
$$\tilde{U}_4 = e^{(i\omega/2)\cos(2u)}\sin u\sum_{n=0}^{\infty}(n+\tfrac{1}{2})b_n^{(4)} \times (-2i\omega x_0\cos^2 u)^n\mathcal{F}(n+1+i\eta, 2n+2; -2i\omega x_0\cos^2 u). \tag{49}$$

3.2.1. Schrödinger equation with Razavy-type potentials. Finkel *et al* [20] have noted that the Schrödinger equation for the Razavy potential [21] is a WHE. This potential belongs to the so-called QES potentials [21–24] for which one part of the energy spectra and the corresponding eigenfunctions can be found exactly. The other portion is supposed to be determined by approximation methods such as perturbation theory or semiclassical methods of approximation [26]. The results below suggest that, for Whittaker–Hill (or Razavy-type) potentials, the whole spectra may be computed by the same methods applicable to the two-centre problem or Teukolsky equations.

Then let us regard the time-independent Schrödinger equation

$$\frac{d^2\psi}{d\xi^2} + [\mathcal{E} - V(\xi)]\psi = 0, \qquad \xi := ax, \qquad \mathcal{E} := \frac{2mE}{\hbar^2 a^2}, \tag{50}$$

where a is a constant and x is the spatial coordinate. For the potential considered by Zaslavskii and Ulyanov [27, 28]

$$V(\xi) = \frac{B^2}{4}\sinh^2\xi - B\left(s + \frac{1}{2}\right)\cosh\xi, \tag{51}$$

where B is a positive constant and s is any non-negative integer or half-integer, the Schrödinger equation is clearly a WHE with $\xi = 2iu$. If we take $x = \cos^2 u = \cosh^2(\xi/2)$, equation (50) reads

$$x(x-1)\frac{d^2\psi}{dx^2} + \left(x - \frac{1}{2}\right)\frac{d\psi}{dx} + \left[\mathcal{E} + B\left(s + \frac{1}{2}\right) + 2B\left(s + \frac{1}{2}\right)(x-1) - B^2x(x-1)\right]\psi = 0,$$

and thus we can choose

$$x_0 = 1, \qquad i\omega = -B, \qquad i\eta = -s - \tfrac{1}{2}, \qquad B_3 = \mathcal{E} + B(s + \tfrac{1}{2}) \tag{52}$$

in the foregoing solutions to the WHE. The the signs for η and ω were chosen so as to satisfy the boundary condition

$$\lim_{\xi\to\pm\infty} \psi = 0. \tag{53}$$

For the present potential, polynomial solutions can be obtained either from the series in hyperbolic functions or in regular confluent hypergeometric functions. The solutions in infinite series are obtained by uniting expansions in series of hyperbolic functions with expansions in series of irregular confluent functions, similar to the case of the radial equation of the two-centre problem. Furthermore, we find that a polynomial solution for s = integer (s = half-integer) corresponds to a pair of matchable expansions (in infinite series) for $s \neq$ integer ($s \neq$ half-integer) and, in particular, for s = half-integer (s = integer). The eigenvalues for infinite-series solutions may be computed as usual, using for example the continued-fraction method [19, 29]. For polynomial solutions the eigenvalues follow from the determinant of a tridiagonal matrix. Indeed, a series with three-term recurrence relations of the type

$$\alpha_0 b_1 + \beta_0 b_0 = 0, \qquad \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \qquad (n \geqslant 1)$$

becomes a finite series with $0 \leqslant n \leqslant N-1$ whenever $\gamma_n = 0$ for $n = N$ [3]. Then the recurrence relations can be written as

$$\begin{pmatrix} \beta_0 & \alpha_0 & 0 & \cdots & & & 0 \\ \gamma_1 & \beta_1 & \alpha_1 & 0 & & & \vdots \\ 0 & \gamma_2 & \beta_2 & \alpha_2 & & & \\ \vdots & & & & & & \\ & & & & \gamma_{N-2} & \beta_{N-2} & \alpha_{N-2} \\ & & & & 0 & \gamma_{N-1} & \beta_{N-1} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ \\ \\ b_{N-2} \\ b_{N-1} \end{pmatrix} = 0 \tag{54}$$

and from this equation we can determine the eigenvalues ($\mathcal{E}$) and the coefficients b_n. For the recurrence relations (32) we must substitute γ_1 by $\gamma_1 + \alpha_{-1}$ in the above matrix and, for (33), we have the replacement $\beta_0 \to \beta_0 + \alpha_{-1}$.

Inserting the parameters (52) into equations (43), (45), (47) and (49), we find the solutions as follows

$$\psi_1 = e^{-(B/2)\cosh\xi} \sum_{n=0}^{\infty} b_n^{(1)} \cosh(n\xi),$$
$$\tilde{\psi}_1 = e^{-(B/2)\cosh\xi} \sum_{n=0}^{\infty} b_n^{(1)} \left(2B\cosh^2\frac{\xi}{2}\right)^n \mathcal{F}\left(n-s, 2n+1; 2B\cosh^2\frac{\xi}{2}\right), \tag{55a}$$

where in the recurrence relations (42*c*) we have

$$\alpha_n^{(1)} = -\frac{B}{2}(n+1+s), \qquad \beta_n^{(1)} = -n^2 - \mathcal{E}, \qquad \gamma_n^{(1)} = \frac{B}{2}(n-s-1). \tag{55b}$$

If s is an integer we get two expressions for polynomial solutions ($\mathcal{F} = (-1)^n\tilde{M}$) with $0 \leqslant n \leqslant s$ seeing that $\gamma_{s+1} = 0$. If s is not an integer (and particularly $s =$ half-integer) we may match the two solutions ($\mathcal{F} = U$) with different regions of convergence to get bounded solutions convergent over the entire range $1 \leqslant x \leqslant \infty$. There are also similar conclusions for the other solutions. Thus, the second pair is

$$\psi_2 = e^{-(B/2)\cosh\xi} \sum_{n=0}^{\infty} b_n^{(2)} \cosh[(n+\tfrac{1}{2})\xi],$$
$$\tilde{\psi}_2 = e^{-(B/2)\cosh\xi} \cosh\frac{\xi}{2} \sum_{n=0}^{\infty} b_n^{(2)} \left(2B\cosh^2\frac{\xi}{2}\right)^n \mathcal{F}\left(n-s+\frac{1}{2}, 2n+2; 2B\cosh^2\frac{\xi}{2}\right), \tag{56a}$$

where in the recurrence relations (44*c*) we have

$$\alpha_n^{(2)} = -\frac{B}{2}\left(n+\frac{3}{2}+s\right), \qquad \beta_n^{(2)} = -(n+\tfrac{1}{2})^2 - \mathcal{E}, \qquad \gamma_n^{(2)} = \frac{B}{2}\left(n-s-\frac{1}{2}\right). \tag{56b}$$

Then, if $s =$ half-integer, we have two expressions for polynomial solutions ($0 \leqslant n \leqslant s - 1/2$) and, if $s \neq$ half-integer, we have a pair of matchable solutions. In the third pair

$$\psi_3 = e^{-(B/2)\cosh\xi} \sum_{n=0}^{\infty} b_n^{(3)} \sinh[(n+1)\xi],$$
$$\tilde{\psi}_3 = e^{-(B/2)\cosh\xi} \sinh\xi \sum_{n=0}^{\infty} (n+1) b_n^{(3)} \left(2B\cosh^2\frac{\xi}{2}\right)^n \mathcal{F}\left(n-s+1, 2n+3; 2B\cosh^2\frac{\xi}{2}\right), \tag{57a}$$

we have

$$\alpha_n^{(3)} = -\frac{B}{2}(n+2+s), \qquad \beta_n^{(3)} = -(n+1)^2 - \mathcal{E}, \qquad \gamma_n^{(3)} = \frac{B}{2}(n-s), \tag{57b}$$

in the recurrence relations (46*c*). If $s =$ integer, we get two expressions for polynomial solutions ($0 \leqslant n \leqslant s-1$) but if $s \neq$ integer we can match the solutions in this pair. The last pair reads

$$\psi_4 = e^{-(B/2)\cosh\xi} \sum_{n=0}^{\infty} b_n^{(4)} \sinh[(n+\tfrac{1}{2})\xi],$$
$$\tilde{\psi}_4 = e^{-(B/2)\cosh\xi} \sinh\frac{\xi}{2} \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right) b_n^{(4)} \times \left(2B\cosh^2\frac{\xi}{2}\right)^n \mathcal{F}\left(n-s+\frac{1}{2}, 2n+2; 2B\cosh^2\frac{\xi}{2}\right), \tag{58a}$$

where in the recurrence relations (48*c*) we have

$$\alpha_n^{(4)} = \alpha_n^{(2)}, \qquad \beta_n^{(4)} = \beta_n^{(2)}, \qquad \gamma_n^{(4)} = \gamma_n^{(2)}, \text{ see equation (56}b\text{)}. \tag{58b}$$

If s = half-integer, both solutions ($\mathcal{F} = (-1)^n \tilde{M}$) are polynomial ($0 \leqslant n \leqslant s - 1/2$) but if $s \neq$ half-integer the series are infinite and we can match them ($\mathcal{F} = U$).

Other Whittaker–Hill potentials can be treated in a similar way. So, the potential investigated by Konwent *et al* [26]

$$V(\xi) = \frac{(2s+1)^2}{4}\left(\frac{B}{2s+1}\cosh\xi - 1\right)^2, \qquad B > 0,\ (s = 0, 1/2, 1, \ldots),$$

can be rewritten as

$$V(\xi) = \frac{B^2}{4}\sinh^2\xi - B\left(s + \frac{1}{2}\right)\cosh\xi + \frac{B^2}{4} + \left(s + \frac{1}{2}\right)^2. \tag{59}$$

The difference of this potential in relation to (51) consists uniquely in a shift in the energy levels, that is, we have just to substitute $\mathcal{E}$ by $\mathcal{E} - B^2/4 - (s + 1/2)^2$ in the previous results. On the other hand, the Razavy potential [21] can be rewritten as

$$V(\xi) = \frac{B^2}{4}\sinh^2(2\xi) - (p+1)B\cosh(2\xi), \qquad B > 0,\ p = 1, 2, 3, \ldots \tag{60a}$$

and the Schrödinger equation is a GSWE (WHE with $u = \mathrm{i}\xi$) characterized by

$$\begin{aligned} &x = \cosh^2\xi, \qquad x_0 = 1, \qquad B_1 = -1/2, \qquad B_2 = 1,\\ &B_3 = [\mathcal{E} + B(2s+2)]/4, \qquad \mathrm{i}\omega = \pm B/2, \qquad \mathrm{i}\eta = \pm(s+1), \end{aligned} \tag{60b}$$

where s was defined by $p = 2s+1$ and then $s = 0, 1/2, 1, 3/2, \ldots$. Inserting these expressions into the solutions to the WHE, we obtain again pairs of infinite-series solutions, in addition to the polynomials solutions obtained by Razavy.

4. Solutions for the confluent GSWE

A confluent GSWE was obtained by Leaver as a limit to the the radial Teukolsky equations for an extreme value for the rotation parameter. More recently an equation, called generalized WHE, has appeared which describes the radial behaviour of a charged massive scalar field on Kerr–Newman spacetimes, in a extreme case as well (see [30], section 4). We can show that the latter is also a confluent GSWE.

For confluent GSWEs the expansions in hypergeometric functions are not valid, but the solution $\tilde{U}_1^{\nu}$ in series of Coulomb wavefunctions affords an appropriate limit. From this limit we get other solutions by the transformations rules t_1 and t_2 and again we arrive at two pairs of solutions with a phase parameter. In section 4.1 we present such solutions and truncate them, and in section 4.2 we discuss some examples.

4.1. The Leaver-type solutions

The first pair is given by

$$\begin{aligned} U_1^{\nu} &= \mathrm{e}^{\mathrm{i}\omega x} x^{-\nu - B_2/2} \sum_{n=-\infty}^{\infty} b_n \left(\frac{B_1}{x}\right)^n \mathcal{F}\left(n + \nu + \frac{B_2}{2}, 2n + 2\nu + 2; \frac{B_1}{x}\right),\\ \tilde{U}_1^{\nu} &= \mathrm{e}^{\mathrm{i}\omega x} x^{\nu + 1 - B_2/2} \sum_{n=-\infty}^{\infty} b_n (-2\mathrm{i}\omega x)^n \mathcal{F}(n + \nu + 1 + \mathrm{i}\eta, 2n + 2\nu + 2; -2\mathrm{i}\omega x), \end{aligned} \tag{61a}$$

where in the recurrence relations

$$\alpha_n = i\omega B_1 \frac{(n+\nu+2-(B_2/2))(n+\nu+1-i\eta)}{2(n+\nu+1)(n+\nu+3/2)},$$
$$\beta_n = B_3 + \left(n+\nu+1-\frac{B_2}{2}\right)\left(n+\nu+\frac{B_2}{2}\right) + \frac{\eta\omega B_1(B_2/2-1)}{(n+\nu)(n+\nu+1)}, \qquad (61b)$$
$$\gamma_n = i\omega B_1 \frac{(n+\nu+(B_2/2)-1)(n+\nu+i\eta)}{2(n+\nu)(n+\nu-1/2)}.$$

These are the Leaver solutions: $\tilde{U}_1^\nu$ is the limit of the corresponding solution in equation (12a) and U_1^ν results from $\tilde{U}_1^\nu$ by the rule t_1. A second pair, obtained by applying the rule t_2 on this first pair, is

$$U_2^\nu = e^{i\omega x+(B_1/x)} x^{-\nu-B_2/2} \sum_{n=-\infty}^{\infty} b'_n \left(-\frac{B_1}{x}\right)^n \mathcal{F}\left(n+\nu+2-\frac{B_2}{2}, 2n+2\nu+2; -\frac{B_1}{x}\right),$$
$$\tilde{U}_2^\nu = e^{i\omega x+(B_1/x)} x^{\nu+1-B_2/2} \sum_{n=-\infty}^{\infty} b'_n (-2i\omega x)^n \mathcal{F}(n+\nu+1+i\eta, 2n+2\nu+2; -2i\omega x), \qquad (62a)$$

where

$$\alpha'_n = i\omega B_1 \frac{(n+\nu+(B_2/2))(n+\nu+1-i\eta)}{2(n+\nu+1)(n+\nu+3/2)}, \qquad \beta'_n = -\beta_n,$$
$$\gamma'_n = i\omega B_1 \frac{(n+\nu+1-(B_2/2))(n+\nu+i\eta)}{2(n+\nu)(n+\nu-1/2)}. \qquad (62b)$$

In the solutions with tilde the series converge for any $|x| > 0$, and in the solutions without tilde the series converge for $|B_1/x| > 0$.

The truncation is similar to the case $x_0 \neq 0$. As a matter of fact, we could obtain the first pair and its recurrence relations starting from the limit to $\tilde{U}_1$ ($x_0 \neq 0$) in equation (35a) and the remaining ones by means of the transformation rules.

First pair. $\nu = B_2/2 - 1$ in $(U_1^\nu, \tilde{U}_1^\nu)$.

$$U_1 = e^{i\omega x} x^{1-B_2} \sum_{n=0}^{\infty} b_n^{(1)} \left(\frac{B_1}{x}\right)^n \mathcal{F}\left(n+B_2-1, 2n+B_2; \frac{B_1}{x}\right),$$
$$\tilde{U}_1 = e^{i\omega x} \sum_{n=0}^{\infty} b_n^{(1)} (-2i\omega x)^n \mathcal{F}\left(n+\frac{B_2}{2}+i\eta, 2n+B_2; -2i\omega x\right), \qquad (63a)$$

$$\alpha_n^{(1)} = i\omega B_1 \frac{(n+1)(n+(B_2/2)-i\eta)}{2(n+(B_2/2))(n+(B_2/2)+1/2)},$$
$$\beta_n^{(1)} = B_3 + n(n+B_2-1) + \frac{\eta\omega B_1((B_2/2)-1)}{(n+(B_2/2)-1)(n+(B_2/2))}, \qquad (63b)$$
$$\gamma_n^{(1)} = i\omega B_1 \frac{(n+B_2-2)(n+(B_2/2)-1+i\eta)}{2(n+(B_2/2)-1)(n+(B_2/2)-3/2)}.$$

Recurrence relations: equation (31) if $B_2 \neq 1, 2$; equation (32) if $B_2 = 1$; equation (33) if $B_2 = 2$.

Second pair. $\nu = 1 - B_2/2$ in $(U_2^\nu, \tilde{U}_2^\nu)$ or $(U_1, \tilde{U}_1) \xrightarrow{t_2} (U_2, \tilde{U}_2)$.

$$U_2 = e^{i\omega x + B_1/x} x^{-1} \sum_{n=0}^{\infty} b_n^{(2)} \left(-\frac{B_1}{x}\right)^n \mathcal{F}\left(n + 3 - B_2, 2n + 4 - B_2; -\frac{B_1}{x}\right),$$
$$\tilde{U}_2 = e^{i\omega x + B_1/x} x^{2-B_2} \sum_{n=0}^{\infty} b_n^{(2)} (-2i\omega x)^n \mathcal{F}\left(n + 2 - \frac{B_2}{2} + i\eta, 2n + 4 - B_2; -2i\omega x\right), \tag{64a}$$

$$\alpha_n^{(2)} = i\omega B_1 \frac{(n+1)(n+2-(B_2/2)-i\eta)}{2(n+2-B_2/2)(n+(5/2)-B_2/2)},$$
$$\beta_n^{(2)} = -B_3 - (n+1)(n+2-B_2) - \frac{\eta\omega B_1((B_2/2)-1)}{(n+1-(B_2/2))(n+2-(B_2/2))}, \tag{64b}$$
$$\gamma_n^{(2)} = i\omega B_1 \frac{(n+2-B_2)(n+1-(B_2/2)+i\eta)}{2(n+1-(B_2/2))(n+(1/2)-(B_2/2))}.$$

Recurrence relations: equation (31) if $B_2 \neq 2, 3$; equation (32) if $B_2 = 3$; equation (33) if $B_2 = 2$.

Third pair: $(U_2, \tilde{U}_2) \xrightarrow{t_1} (\tilde{U}_3, U_3)$.

$$U_3 = e^{-i\omega x} x^{i\eta - B_2/2} \sum_{n=0}^{\infty} b_n^{(3)} \left(\frac{B_1}{x}\right)^n \mathcal{F}\left(n - i\eta + \frac{B_2}{2}, 2n + 2 - 2i\eta; \frac{B_1}{x}\right),$$
$$\tilde{U}_3 = e^{-i\omega x} x^{1-i\eta-B_2/2} \sum_{n=0}^{\infty} b_n^{(3)} (2i\omega x)^n \mathcal{F}(n + 1 - 2i\eta, 2n + 2 - 2i\eta; 2i\omega x), \tag{65a}$$

$$\alpha_n^{(3)} = i\omega B_1 \frac{(n+1)(n+2-i\eta-(B_2/2))}{2(n+1-i\eta)(n-i\eta+(3/2))},$$
$$\beta_n^{(3)} = -B_3 - \left(n + 1 - i\eta - \frac{B_2}{2}\right)\left(n - i\eta + \frac{B_2}{2}\right) - \frac{\eta\omega B_1((B_2/2)-1)}{(n-i\eta)(n+1-i\eta)}, \tag{65b}$$
$$\gamma_n^{(3)} = i\omega B_1 \frac{(n-2i\eta)(n+(B_2/2)-i\eta-1)}{2(n-i\eta)(n-i\eta-1/2)}.$$

Recurrence relations: equation (31) if $i\eta \neq 0, 1/2$; equation (32) if $i\eta = 1/2$; equation (33) if $i\eta = 0$.

These solutions may also be derived by taking $\nu = i\eta$ in $(U_1^\nu, \tilde{U}_1^\nu)$ and then using the rule T_3.

Fourth pair. $(U_3, \tilde{U}_3) \xrightarrow{t_2} (U_4, \tilde{U}_4)$.

$$U_4 = e^{-i\omega x + B_1/x} x^{i\eta - (B_2/2)} \sum_{n=0}^{\infty} b_n^{(4)} \left(-\frac{B_1}{x}\right)^n \mathcal{F}\left(n + 2 - i\eta - \frac{B_2}{2}, 2n + 2 - 2i\eta; -\frac{B_1}{x}\right),$$
$$\tilde{U}_4 = e^{-i\omega x + (B_1/x)} x^{1-i\eta-B_2/2} \sum_{n=0}^{\infty} b_n^{(4)} (2i\omega x)^n \mathcal{F}(n + 1 - 2i\eta, 2n + 2 - 2i\eta; 2i\omega x), \tag{66a}$$

$$\alpha_n^{(4)} = -i\omega B_1 \frac{(n+1)(n+(B_2/2)-i\eta)}{2(n+1-i\eta)(n-i\eta+3/2)},$$
$$\beta_n^{(4)} = -B_3 - \left(n + 1 - i\eta - \frac{B_2}{2}\right)\left(n - i\eta + \frac{B_2}{2}\right) - \frac{\eta\omega B_1((B_2/2)-1)}{(n-i\eta)(n+1-i\eta)}, \tag{66b}$$
$$\gamma_n^{(4)} = -i\omega B_1 \frac{(n-2i\eta)(n-(B_2/2)-i\eta+1)}{2(n-i\eta)(n-i\eta-1/2)}.$$

Recurrence relations: equation (31) if $i\eta \neq 0, 1/2$; equation (32) if $i\eta = 1/2$; equation (33) if $i\eta = 0$.

These solutions may be obtained, if we prefer, putting $\nu = i\eta$ into $(U_2^\nu, \tilde{U}_2^\nu)$ and then using the rule T_3.

4.2. Examples

As examples we discuss the time dependence of a massive test fermion in nonflat dust-dominated FRW universe models (there is no free parameter in the differential equation) and the Schrödinger equation for QES asymmetric double-Morse potentials (the energy represents a free parameter).

4.2.1. Dirac equation in dust-dominated FRW spacetimes. For FRW universes filled with dust the scale factor is given by $A(t) = a_0[1-\cos(\sqrt{\epsilon}\tau)]/\epsilon$. So, equation (25) for $S(x)$ reads

$$\frac{d^2S}{d\tau^2} + \left[\sigma^2 + i\mu a_0 \frac{\sin(\sqrt{\epsilon}\tau)}{\sqrt{\epsilon}} + \mu^2 a_0^2[1-\cos(\sqrt{\epsilon}\tau)]^2\right]S = 0, \tag{67}$$

which can be reduced to a confluent GSWE. In effect, the change of variable

$$x = e^{-i\sqrt{\epsilon}\tau} = \cos\sqrt{\epsilon}\tau - i\sin\sqrt{\epsilon}\tau \tag{68a}$$

gives

$$x^2\frac{d^2S}{dx^2} + x\frac{dS}{dx} + \left[k - \frac{A_1}{x^2} - \frac{A_2}{x} - A_3x - A_4x^2\right]S = 0, \tag{68b}$$

where

$$\begin{aligned} k &= -\epsilon(\sigma^2 + \tfrac{3}{2}\mu^2 a_0^2), & A_1 &= A_4 = \tfrac{1}{4}\epsilon\mu^2 a_0^2, \\ A_2 &= -\epsilon\mu^2 a_0^2 + \frac{\sqrt{\epsilon}}{2}\mu a_0, & A_3 &= -\epsilon\mu^2 a_0^2 - \frac{\sqrt{\epsilon}}{2}\mu a_0. \end{aligned} \tag{68c}$$

The substitution

$$S(x) = e^{a/x}x^bU(x), \qquad a^2 := A_1, \qquad a - 2ab - A_2 := 0, \tag{69a}$$

furnishes

$$x^2\frac{d^2U}{dx^2} + [(2b+1)x - 2a]\frac{dU}{dx} + [-A_4x^2 - A_3x + k + b^2]U = 0, \tag{69b}$$

that is, a confluent GSWE with

$$B_1 = -2a, \qquad B_2 = 2b+1, \qquad B_3 = k + b^2, \qquad \omega^2 = -A_4 \text{ e } 2\eta\omega = A_3,$$

or, choosing $a = \sqrt{A_1} = \mu a_0\sqrt{\epsilon}/2$,

$$\begin{aligned} B_1 &= -\mu a_0\sqrt{\epsilon}, & B_2 &= 1 + 2\mu a_0\sqrt{\epsilon}, & B_3 &= -\epsilon(\sigma^2 + \tfrac{1}{2}\mu^2 a_0^2), \\ i\omega &= \pm\frac{\mu a_0}{2}\sqrt{\epsilon}, & i\eta &= \pm(\tfrac{1}{2} + \mu a_0\sqrt{\epsilon}). \end{aligned} \tag{69c}$$

Therefore, the solutions for $S(x)$ may be obtained by means of

$$S_i^\nu(x) = e^{-B_1/(2x)}x^{(B_2-1)/2}U_i^\nu(x), \tag{70}$$

where $U_i^{\nu}(x)$ denotes the expansions with phase parameters given in section 4.1. Explicitly we have

$$S_1^{\nu} = \mathrm{e}^{\mathrm{i}\omega x-(B_1/2x)}x^{-\nu-1/2}\sum_{n=-\infty}^{\infty} b_n\left(\frac{B_1}{x}\right)^n\mathcal{F}\left(n+\nu+\frac{B_2}{2}, 2n+2\nu+2; \frac{B_1}{x}\right),$$
$$\tilde{S}_1^{\nu} = \mathrm{e}^{\mathrm{i}\omega x-(B_1/2x)}x^{\nu+1/2}\sum_{n=-\infty}^{\infty} b_n(-2\mathrm{i}\omega x)^n\mathcal{F}(n+\nu+1+\mathrm{i}\eta, 2n+2\nu+2; -2\mathrm{i}\omega x), \qquad (71a)$$

$$S_2^{\nu} = \mathrm{e}^{\mathrm{i}\omega x+(B_1/2x)}x^{-\nu-1/2}\sum_{n=-\infty}^{\infty} b'_n\left(-\frac{B_1}{x}\right)^n\mathcal{F}\left(n+\nu+2-\frac{B_2}{2}, 2n+2\nu+2; -\frac{B_1}{x}\right),$$
$$\tilde{S}_2^{\nu} = \mathrm{e}^{\mathrm{i}\omega x+(B_1/2x)}x^{\nu+1/2}\sum_{n=-\infty}^{\infty} b'_n(-2\mathrm{i}\omega x)^n\mathcal{F}(n+\nu+1+\mathrm{i}\eta, 2n+2\nu+2; -2\mathrm{i}\omega x). \qquad (71b)$$

For $\epsilon = 1$ we have $|x| = |\mathrm{e}^{\mathrm{i}\tau}| = 1$ and accordingly there is no problem about series convergence or regularity condition; in this case we must choose one solution of each pair. Also for $\epsilon = -1$ we do not have problems with respect to convergence or regularity condition as long as we match the solutions of each pair since now $0 \leqslant |x| = |\mathrm{e}^{\tau}| < \infty$.

We note here that the radial equation for the scalar field mentioned at the beginning of this section 4 (called the generalized WHE) is

$$x^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}R}{\mathrm{d}x} + \left[(\omega^2-\mu^2)M^2x^2 + 2(A\omega - M\mu^2)Mx + \left(A+\frac{\mathcal{B}}{x}\right)^2 + (2\omega-\mu^2)(2M^2-e^2) - 2qeM\omega - \lambda\right]R = 0, \qquad (72)$$

where the constants are defined in the paper by Wu and Cai [30]. Since it has the same form as equation (68*b*), we may reduce it to a confluent GSWE, as we have stated elsewhere.

4.2.2. Schrödinger equation with asymmetric double-Morse potentials. We consider the Schrödinger equation (50) for QES asymmetric double-Morse potentials. Contrary to the case of the (symmetric) Razavy-type potentials of section 3.2.1, we find that it is not possible to match solutions belonging to the same pair of solutions in order to get infinite-series solutions convergent and bounded for the entire range of the independent variable. Even for polynomial solutions there are some problems.

Let us consider the Turbiner generalized Morse potential [22, 23], whose form is

$$V(\xi) = k + A_1\mathrm{e}^{-2\xi} + A_2\mathrm{e}^{-\xi} + A_3\mathrm{e}^{\xi} + A_4\mathrm{e}^{2\xi}, \qquad (73a)$$

where we suppose that A_1 and A_2 are positive and $A_2 \neq \pm A_3$. In analogy with the case of dust-dominated FRW spacetimes, we perform the substitutions

$$x = \mathrm{e}^{\xi}, \qquad \psi(\xi) = \mathrm{e}^{a/x}x^bU(x), \qquad a^2 = A_1, \qquad a - 2ab - A_2 = 0, \qquad (73b)$$

which reduce the Schrödinger equation to

$$x^2\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + [(2b+1)x - 2a]\frac{\mathrm{d}U}{\mathrm{d}x} + [-A_4x^2 - A_3x + b^2 + \mathcal{E} - k]U = 0, \qquad (73c)$$

that is, to a confluent GSWE having

$$B_1 = -2a, \qquad B_2 = 2b+1, \qquad B_3 = \mathcal{E} + b^2 - k,$$
$$\omega^2 = -A_4 \text{ e } 2\eta\omega = A_3. \qquad (73d)$$

Therefore the solutions must present the same form as in the previous example, namely,

$$\psi_i = \mathrm{e}^{-B_1/(2x)}x^{(B_2-1)/2}U_i(x), \qquad (74)$$

but now $U_i(x)$ denotes the four pairs of solutions without phase parameters.

Now let

$$V(\xi) = \frac{B^2}{4}\left(\sinh\xi - \frac{C}{B}\right)^2 - B\left(s + \frac{1}{2}\right)\cosh\xi; \qquad s = 0, 1/2, 1, 3/2\dots. \tag{75}$$

be the asymmetric double-Morse potential considered by Zaslavskii and Ulyanov [27, 28], where $B > 0$, $C > 0$. It can be written as the exponential potential (73*a*) with

$$k = \frac{C^2}{4} - \frac{B^2}{8}, \qquad A_1 = A_4 = \frac{B^2}{16},$$
$$A_2 = \frac{B}{4}(C - 2s - 1), \qquad A_3 = -\frac{B}{4}(C + 2s + 1).$$

The choices $a = -\sqrt{A_1} = -B/4$ and $\omega = \mathrm{i}\sqrt{A_4} = \mathrm{i}B/4$ yield the parameters

$$B_1 = \frac{B}{2}, \qquad B_2 = 1 + C - 2s, \qquad B_3 = \mathcal{E} + \frac{B^2}{8} + s^2 - sC,$$
$$\mathrm{i}\omega = -\frac{B}{4}, \qquad \mathrm{i}\eta = -\frac{C}{2} - \frac{1}{2} - s \tag{76}$$

for the confluent GSWE. Then, using the first pair of wavefunctions, we obtain

$$\psi_1 = \mathrm{e}^{-(B/2)\cosh\xi - ((C/2)-s)\xi}\sum_{n=0}^{\infty} b_n^{(1)}\left(\frac{B}{2}\mathrm{e}^{-\xi}\right)^n \mathcal{F}\left(n + C - 2s, 2n + C + 1 - 2s; \frac{B}{2}\mathrm{e}^{-\xi}\right),$$
$$\tilde{\psi}_1 = \mathrm{e}^{-(B/2)\cosh\xi + ((C/2)-s)\xi}\sum_{n=0}^{\infty} b_n^{(1)}\left(\frac{B}{2}\mathrm{e}^{\xi}\right)^n \mathcal{F}\left(n - 2s, 2n + C + 1 - 2s; \frac{B}{2}\mathrm{e}^{\xi}\right), \tag{77}$$

with recurrence relations given by equation (31) if $C \neq 2s$ or $2s + 1$, equation (32) if $C = 2s$, equation (33) if $C = 2s + 1$ and having the coefficients

$$\alpha_n^{(1)} = \frac{B^2}{16}\frac{(n+1)(n+C+1)}{(n + (C/2) + (1/2) - s)(n + (C/2) + 1 - s)},$$
$$\beta_n^{(1)} = -\mathcal{E} - s(s - C) - \frac{B^2}{8} - n(n + C - 2s) + \frac{B^2[C^2 - (1+2s)^2]}{32(n + (C/2) - (1/2) - s)(n + (C/2) + (1/2) - s)}, \tag{78}$$
$$\gamma_n^{(1)} = \frac{B^2}{16}\frac{(n + C - 2s - 1)(n - 2s - 1)}{(n + (C/2) - (1/2) - s)(n + (C/2) - 1 - s)}.$$

If s is a non-negative integer or half-integer, we have $\gamma_{2s+1} = 0$, and therefore $\tilde{\psi}_1$ with $\mathcal{F} = (-1)^n\tilde{M}$ is a polynomial solution with n running from 0 to $2s$. This solution holds only when $C \neq$ integer or $C =$ integer $\geqslant 2s$; for $C =$ integer $< 2s$, the regular confluent hypergeometric functions are not defined. The eigenvalues and the expansions coefficients can be determined from equation (54). On the other hand, if s is not a non-negative integer or half-integer both the solutions in (77), with regular or irregular confluent hypergeometric functions, can be combined to give convergent and bounded solutions in terms of infinite series.

For $s =$ integer or half-integer, infinite-series wavefunctions bounded for all values of ξ cannot be obtained by matching solutions belonging to the same pair. Such solutions would need to present the factor $\exp(-(B/2)\cosh\xi)$ but this does not happen. Thus, from the second pair we obtain

$$\psi_2 = \mathrm{e}^{-(B/2)\sinh\xi+((C/2)-s-1)\xi}\sum_{n=0}^{\infty} b_n^{(2)} \times \left(\frac{B}{2}\mathrm{e}^{-\xi}\right)^n \tilde{M}\left(n+2+2s-C, 2n+3+2s-C; -\frac{B}{2}\mathrm{e}^{-\xi}\right), \tag{79a}$$

$$\tilde{\psi}_2 = \mathrm{e}^{-(B/2)\sinh\xi-((C/2)-s-1)\xi}\sum_{n=0}^{\infty} b_n^{(2)}\left(\frac{B}{2}\mathrm{e}^{\xi}\right)^n U\left(n+1-C, 2n+3+2s-C; \frac{B}{2}\mathrm{e}^{\xi}\right),$$

where, in the recurrence relations, we have

$$\alpha_n^{(2)} = \frac{B^2(n+1)(n+2+2s)}{16(n+s-(C/2)+(3/2)+s)(n-(C/2)+2+s)},$$
$$\beta_n^{(2)} = \mathcal{E} + s(s-C) + \frac{B^2}{8} + (n+1)(n+1-C+2s) - \frac{B^2[C^2-(1+2s)^2]}{32(n-(C/2)+(1/2)+s)(n-(C/2)+(3/2)+s)}, \tag{79b}$$
$$\gamma_n^{(2)} = \frac{B^2(n-C+2s+1)(n-C)}{16(n-(C/2)+(1/2)+s)(n-(C/2)+s)}.$$

In these two solutions we have infinite series (if $C \neq$ integer) but the solutions are unbounded when $\xi \to -\infty$. If $C =$ integer, the solution ψ_2 is polynomial but unbounded when $\xi \to -\infty$. Similarly, from the third pair, we get

$$\psi_3 = \mathrm{e}^{(B/2)\sinh\xi-((C/2)+s+1)\xi}\sum_{n=0}^{\infty} b_n^{(3)}\left(\frac{B}{2}\mathrm{e}^{-\xi}\right)^n U\left(n+1+C, 2n+3+2s+C; \frac{B}{2}\mathrm{e}^{-\xi}\right),$$
$$\tilde{\psi}_3 = \mathrm{e}^{(B/2)\sinh\xi+((C/2)+s+1)\xi}\sum_{n=0}^{\infty} b_n^{(3)}\left(\frac{B}{2}\mathrm{e}^{\xi}\right)^n \tilde{M}\left(n+2+2s+C, 2n+3+2s+C; -\frac{B}{2}\mathrm{e}^{\xi}\right), \tag{80a}$$

where, in the recurrence relations, we have

$$\alpha_n^{(3)}(C,s) = \alpha_n^{(2)}(-C,s), \qquad \beta_n^{(3)}(C,s) = \beta_n^{(2)}(-C,s), \qquad \gamma_n^{(3)}(C,s) = \gamma_n^{(2)}(-C,s). \tag{80b}$$

Both solutions are again given by infinite series, but now they are unbounded when $\xi \to \infty$. Note that, for $C \neq$ integer, we could match solutions taking from the second and third pairs but it would be necessary to show that, in both cases, each one with a different characteristic equation, the eigenvalues converge to the same limit. It would be better to seek new solutions for this problem. The same occurs with other potentials as, for example, the asymmetric potential studied by Konwent *et al*

$$V(\xi) = \frac{(2s+1)^2}{4}\left(\frac{B}{2s+1}\cosh\xi - 1\right)^2 + \frac{BC}{2}\sinh\xi; \qquad CeB > 0;\ s = 0, 1/2, 1, 3/2, \ldots,$$

or the potential [24]

$$V(\xi) = \delta^2\mathrm{e}^{-2\xi} + 2\delta(\gamma-1)\mathrm{e}^{-\xi} - 2\beta(p+\gamma)\mathrm{e}^{\xi} + \beta^2\mathrm{e}^{2\xi}; \qquad p = 0, 1, 2, \ldots, \tag{81a}$$

where we suppose that δ and β are positive and $\delta \neq \beta$. Thus, if in the latter case we select $a = -\sqrt{A_1} = -\delta$ and $\mathrm{i}\omega = -\sqrt{A_4} = -\beta$, we obtain

$$B_1 = 2\delta, \qquad B_2 = 2\gamma, \qquad B_3 = \mathcal{E} + \left(\gamma - \tfrac{1}{2}\right)^2, \qquad \mathrm{i}\omega = -\beta, \qquad \mathrm{i}\eta = -\gamma - p. \tag{81b}$$

Then, the first pair of solutions provides

$$\psi_1 = f_1^+(x)\sum_{n=0}^{\infty} b_n^{(1)}(2\delta e^{-\xi})^n \mathcal{F}(n+2\gamma-1, 2n+2\gamma; 2\delta e^{-\xi}),$$
$$\tilde{\psi}_1 = f_1^-(x)\sum_{n=0}^{\infty} b_n^{(1)}(2\beta e^{\xi})^n \mathcal{F}(n-p, 2n+2\gamma; 2\beta e^{\xi}), \qquad (82a)$$
$$f_1^{\pm}(x) := \exp\left[-\beta e^{\xi} - \delta e^{-\xi} \pm \tfrac{1}{2}(1-2\gamma)\xi\right]. \qquad (82b)$$

$$\alpha_n^{(1)} = -\frac{\beta\delta(n+1)(n+2\gamma+p)}{(n+\gamma)(n+\gamma+1/2)},$$
$$\beta_n^{(1)} = \mathcal{E} + \left(\gamma - \frac{1}{2}\right)^2 + n(n+2\gamma-1) - \frac{2\beta\delta(\gamma+p)(\gamma-1)}{(n+\gamma-1)(n+\gamma)}, \qquad (82c)$$
$$\gamma_n^{(1)} = -\frac{\beta\delta(n+2\gamma-2)(n-p-1)}{(n+\gamma-1)(n+\gamma-3/2)},$$

with recurrence relations given by equation (31) if $2\gamma \neq 1, 2$, equation (32) if $2\gamma = 1$, and equation (33) if $\gamma = 1$.

If p is a non-negative integer we have $\gamma_{p+1} = 0$, and therefore the solution $\tilde{\psi}_1$ with $\mathcal{F} = (-1)^n \tilde{M}$ is a regular polynomial solution with n extending from 0 to p. However, if 2γ is zero or a negative integer, the regular hypergeometric function is not well defined. On the other hand, if p is not a non-negative integer the solutions in (82a), both with regular or irregular confluent hypergeometric functions, can be matched to give convergent and bounded solutions in terms of infinite series, but only when 2γ is not zero or a negative integer. Moreover, using the second and third pairs of solutions we may verify that (for p = integer) infinite-series wavefunctions bounded for $\xi \in (-\infty, \infty)$ cannot again be obtained by matching solutions belonging to the same pair.

Finally, note that we have seen that the Schrödinger equation for the potential (73a) is analogous to the Dirac equation (67) with $\epsilon = -1$. There is also an analogue for $\epsilon = 1$, given by a periodic QES potential whose form is [31]

$$V(\xi) = A\cos(2\xi) + B\cos\xi + C\sin\xi + D\sin(2\xi), \qquad (83a)$$

that can be rewritten as

$$V(\xi) = A_1 e^{-2i\xi} + A_2 e^{-i\xi} + A_3 e^{i\xi} + A_4 e^{2i\xi},$$
$$A_1 := \tfrac{1}{2}(A + iD), \qquad A_2 := \tfrac{1}{2}(B + iC), \qquad A_3 := A_2^*, \qquad A_4 := A_1^*. \qquad (83b)$$

Indeed, the changes of variables

$$x = e^{i\xi}, \qquad \psi(\xi) = e^{a/x} x^b U(x); \qquad a^2 = -A_1, \qquad a - 2ab + A_2 = 0, \qquad (84a)$$

in the Schrödinger equation imply that U is ruled by

$$x^2 \frac{d^2 U}{dx^2} + [(2b+1)x - 2a]\frac{dU}{dx} + [A_4 x^2 + A_3 x + b^2 - \mathcal{E}]U = 0, \qquad (84b)$$

that is, by a confluent GSWE in which

$$B_1 = -2a, \qquad B_2 = 2b+1, \qquad B_3 = b^2 - \mathcal{E}, \qquad \omega^2 = A_4, \qquad 2\eta\omega = -A_3. \qquad (84c)$$

5. Final remarks

The solutions to the GSWEs presented in this paper have been developed according to the principles exposed in section 1. In section 2, expansions with phase parameter have been written as two pairs of solutions, each one having the same series coefficients and consisting of a solution in series of hypergeometric functions, and a second one in series of Coulomb wavefunctions. The first solution converges in any finite region of the complex plane, while the second converges for $|x| > |x_0|$. For the WHE, the series in hypergeometric functions reduce to even or odd series of trigonometric (or hyperbolic) functions with a counterpart in series of Coulomb wavefunctions. Equations for the time dependence of the Dirac test fermions in nonflat radiation-dominated FRW spacetimes have been tranformed into Whittaker–Hill-type equations in which all the constants are known.

In section 3 we have supposed that there is some free parameter in the GSWE, and then the expansions found in section 2 were truncated, giving four pairs of solutions without phase parameter. The truncation of the series in hypergeometric functions provided solutions of the Fackerell–Crosmann type, that is, in series of Jacobi polynomials. Given one pair of solutions, the others can be generated by means of the transformations rules T_1 and T_2. For the angular two-centre problem, solutions in series of regular Coulomb wavefunctions were established, in addition to the Baber–Hassé expansions in series of associated Legendre polynomials. Analogously, for the angular Teukolsky equations, solutions in series of regular Coulomb wavefunctions were obtained, in addition to the Fackerell–Crossman expansions. For the radial two-centre problem, solutions bounded over the entire range of the radial variable were found by matching expansions in series of irregular Coulomb wavefunctions with expansions in series of hypergeometric functions. This procedure offers computational advantages in relation to that used by Liu [19], since the matchable solutions are given in terms of one-sided series without phase parameters and both solutions have the same eigenvalue equation.

Still, in section 3, the four Arscott solutions in series of trigonometric (or hyperbolic) functions were recovered, and each of them corresponds to an expansion in series of Coulomb wavefunctions. They were applied to formally solve the Schrödinger equation with Razavy-type potentials. Polynomial solutions in series of hyperbolic functions and regular Coulomb wavefunctions were found. Solutions in infinite series were composed by connecting expansions in series of hyperbolic functions with expansions in series of irregular Coulomb wavefunctions, similar to the case of the radial two-centre problem. These solutions in infinite series seem to be suitable to find the complete energy spectrum without using the common approximation methods.

To consider the WHE as a special GSWE is not a novelty (see part B of [4]). However, one has the impression that so far this information has not been used to derive explicit solutions to the WHE, as we have done in sections 2.2 and 3.2. The prescription for this is as follows: find a solution for the GSWE in its general form (1), use the transformations rules given in section 1, and then particularize the solutions to the WHE.

In section 4, we have used the transformation rule t_2 to generalize the Leaver solutions in series of Coulomb functions for confluent GSWEs. We have shown that these solutions may be used to find the time dependence of massive Dirac test fields in dust-dominated FRW spacetimes. The truncated solutions were applied to get polynomial solutions to the Schrödinger equation with QES asymmetric double-Morse potentials. In this case no satisfactory infinite-series solution were found, and the search of new solutions appropriate for the case remains open. Note the new instances of confluent GSWE that were found in this section: the Schrödinger equation for the potentials (73a) and (83a), the equation (67) for the time dependence of a Dirac field in dust-dominated FRW backgrounds, and the equation (72) for the radial dependence of a massive scalar field in Kerr–Newman spacetimes.

In the appendix we have derived the recurrence relations for the truncated expansions with $x_0 \neq 0$. We have obtained three possibly different recurrence relations, each of them being valid for the solutions of the WHE.

Throughout the text we have taken several equations from mathematical physics as mere examples. This is particularly true with respect to the equations for the time dependence of Dirac test fermions in FRW backgrounds inasmuch as we have not written explicitly the solutions for $S(\tau)$ and $T(\tau)$. To solve these equations, in addition to considering the regularity and convergence conditions, we need to find four independent sets of solutions and check if they satisfy the requirements of 'charge conjugation', since the Dirac equation in FRW spacetimes is invariant under such an operation.

We have not examined the integral relationships which may exist between solutions with the same recurrence relations either. In effect, Masuda and Susuki [11] found that Otchik-type solutions in series of hypergeometric and Coulomb wavefunctions are related by means of integral transformations. Thus, we can extend that study to the generalized solutions investigated here and, in particular, to the truncated solutions. This extension might also include solutions of Jaffè and Hilleraas type for which Leaver found integral relations only for special values of the parameter η [1].

Another open issue concerns the generalization of the expansions in series of Coulomb wavefunctions to a Heun differential equation in its general form, as well as the possibility of getting pairs constituted by such expansions and expansions in hypergeometric functions, as in the case of GSWEs. Actually, we know that there are QES potentials which lead to general Heun equations [32] and, if that generalization is possible, perhaps we could also find infinite-series solutions appropriate for these problems. A further question refers to the connections between the Schrödinger equation for other QES potentials and the Heun equation or its special cases. We advance that, for the trigonometric and hyperbolic potentials of [24,31], the Schrödinger equation may be transformed into GSWEs, and for this reason it has the pairs of solutions found in section 3.1 as candidates for polynomial and infinite-series solutions. Nevertheless, it is also necessary to consider other classes of QES potentials.

Appendix A. Truncation and recurrence relations

Let us see how we have obtained the first pair of solutions $(U_1, \tilde{U}_1)$. For $n \geqslant 0$ the solution U_1^ν reads

$$U_1^\nu = \mathrm{e}^{\mathrm{i}\omega x} \sum_{n=0}^{\infty} b_n F\left(\frac{B_2}{2} - n - \nu - 1, n + \nu + \frac{B_2}{2}; B_2 + \frac{B_1}{x_0}; \frac{x - x_0}{x_0}\right), \quad \text{(A.1)}$$

which, when inserted into equation (1), gives

$$\begin{aligned}
&\alpha_{-1} b_0 F\left(\frac{B_2}{2} - \nu, \frac{B_2}{2} + \nu - 1; B_2 + \frac{B_1}{x_0}; y\right) \\
&\quad + (\alpha_0 b_1 + \beta_0 b_0) F\left(\frac{B_2}{2} - \nu - 1, \frac{B_2}{2} + \nu; B_2 + \frac{B_1}{x_0}; y\right) \\
&\quad + (\alpha_1 b_2 + \beta_1 b_1 + \gamma_1 b_0) F\left(\frac{B_2}{2} - \nu - 2, \frac{B_2}{2} + \nu + 1; B_2 + \frac{B_1}{x_0}; y\right) \\
&\quad + \sum_{n=2}^{\infty} (\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1}) \\
&\quad \times F\left(\frac{B_2}{2} - n - \nu - 1, n + \nu + \frac{B_2}{2}; B_2 + \frac{B_1}{x_0}; y\right) = 0 \qquad \text{(A.2)}
\end{aligned}$$

where $y = (x_0 - x)/x_0$. The parameter ν must be chosen so that the coefficients of each independent term vanish. Whenever $\alpha_{-1} = 0$ we have the recurrence relations (31) but there are cases in which α_{-1} is not zero, as the right-hand side of the following expression suggests

$$\frac{\alpha_{-1}}{\mathrm{i}\omega x_0} = \frac{(\nu + 1 - (B_2/2))(\nu - (B_1/x_0) - (B_2/2))(\nu - \mathrm{i}\eta)}{2\nu(\nu + 1/2)} :$$

$$\begin{cases} \nu = (B_2/2) - 1; & \alpha_{-1} = 0 \quad \text{if } B_2 \neq 1, 2; \\ \nu = (B_1/x_0) + (B_2/2); & \alpha_{-1} = 0 \quad \text{if } (B_1/x_0) + (B_2/2) \neq 0, \frac{1}{2}; \\ \nu = \mathrm{i}\eta; & \alpha_{-1} = 0 \quad \text{if } \mathrm{i}\eta \neq 0, -\frac{1}{2}. \end{cases}$$

In effect we see that there are three possible choices for ν and in each of them there are two exceptions for which α_{-1} may not vanish. Hereafter we discard the possibility $\nu = \mathrm{i}\eta$ because it does not lead to solutions in terms of Jacobi's polynomials. For the exceptions we find two dependent terms in equation (A.2). Considering the possibility $\nu = B_2/2 - 1$, we obtain the solution U_1 with the recurrence relations (31), when $B_2 \neq 1, 2$. If $B_2 = 1$ ($\nu = -1/2$), equation (A.2) becomes

$$\alpha_{-1} b_0 F\left(1, -1; 1 + \frac{B_1}{x_0}; y\right) + (\alpha_0 b_1 + \beta_0 b_0) F\left(0, 0; 1 + \frac{B_1}{x_0}; y\right) + (\alpha_1 b_2 + \beta_1 b_1 + \gamma_1 b_0) F\left(-1, 1; 1 + \frac{B_1}{x_0}; y\right) + \cdots = 0.$$

As the first and the third terms are linearly dependent, we get the recurrence relations (32). On the other hand, if $B_2 = 2$ ($\nu = 0$) we have

$$\alpha_{-1} b_0 F\left(1, 0; 2 + \frac{B_1}{x_0}; y\right) + (\alpha_0 b_1 + \beta_0 b_0) F\left(0, 1; 2 + \frac{B_1}{x_0}; y\right) + (\alpha_1 b_2 + \beta_1 b_1 + \gamma_1 b_0) F\left(-1, 2; 2 + \frac{B_1}{x_0}; y\right) + \cdots = 0.$$

Since the first and the second terms are constant, the recurrence relations have the form given in (33). Therefore, we have derived the solution U_1. Now let us consider the solution $\tilde{U}_1^{\nu}$ for $n \geqslant 0$

$$\tilde{U}_1{}^{\nu} = \mathrm{e}^{\mathrm{i}\omega x}(x - x_0)^{\nu+1-(B_2/2)} \sum_{n=0}^{\infty} \tilde{b}_n (-2\mathrm{i}\omega x)^n \mathcal{F}(n + \nu + 1 + \mathrm{i}\eta, 2n + 2\nu + 2; -2\mathrm{i}\omega x). \quad \text{(A.3)}$$

If $\mathcal{F}(a_n, b_n; z) = U(a_n, b_n; z)$ we get

$$\alpha_{-1}(-2\mathrm{i}\omega x)^{-1} b_0 U(\nu + \mathrm{i}\eta, 2\nu; -2\mathrm{i}\omega x) + (\alpha_0 b_1 + \beta_0 b_0) U(\nu + 1 + \mathrm{i}\eta, 2\nu + 2; -2\mathrm{i}\omega x) + (\alpha_1 b_2 + \beta_1 b_1 + \gamma_1 b_0)(-2\mathrm{i}\omega x) U(\nu + 2 + \mathrm{i}\eta, 2\nu + 4; -2\mathrm{i}\omega x) + \sum_{n=2}^{\infty} (\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1})(-2\mathrm{i}\omega x)^n \times U(n + \nu + 1 + \mathrm{i}\eta, 2n + 2\nu + 2; -2\mathrm{i}\omega x) = 0. \quad \text{(A.4)}$$

In order to obtain the solution $\tilde{U}_1$, the counterpart for U_1, we again choose $\nu = B_2/2 - 1$. To find the recurrence relations for $B_2 = 1$ and $B_2 = 2$ we use [12]

$$U(a, 1 - n; z) = z^n U(a + n, 1 + n, z)$$

that implies

$$U(\mathrm{i}\eta - \tfrac{1}{2}, -1; -2\mathrm{i}\omega x) = (2\mathrm{i}\omega x)^2 U(\mathrm{i}\eta + \tfrac{3}{2}, 3; -2\mathrm{i}\omega x),$$
$$U(\mathrm{i}\eta, 0; -2\mathrm{i}\omega x) = -2\mathrm{i}\omega x U(\mathrm{i}\eta + 1, 2; -2\mathrm{i}\omega x).$$

Then, for $B_2 = 1$ ($\nu = -1/2$) we have

$$\alpha_{-1}(-2\mathrm{i}\omega x)^{-1} b_0 U(\mathrm{i}\eta - \tfrac{1}{2}, -1; -2\mathrm{i}\omega x) + (\alpha_0 b_1 + \beta_0 b_0) U(\mathrm{i}\eta + \tfrac{1}{2}, 1; -2\mathrm{i}\omega x) + (\alpha_1 b_2 + \beta_1 b_1 + \gamma_1 b_0)(-2\mathrm{i}\omega x) U(\mathrm{i}\eta + \tfrac{3}{2}, 3; -2\mathrm{i}\omega x) + \cdots = 0,$$

and the first and the third terms are linearly dependent giving the equation (32). For $B_2 = 2$ ($\nu = 0$) we get

$$\alpha_{-1}(-2i\omega x)^{-1}b_0U(i\eta, 0; -2i\omega x) + (\alpha_0b_1 + \beta_0b_0)U(1 + i\eta, 2; -2i\omega x) + (\alpha_1b_2 + \beta_1b_1 + \gamma_1b_0)(-2i\omega x)U(2 + i\eta, 4; -2i\omega x) + \cdots = 0,$$

and we see that the first and the second terms are linearly dependent; this leads to the recurrence relations given by equation (33). To complete the derivation of the pair $(U_1, \tilde{U}_1)$ we must still suppose that $\mathcal{F}(a_n, b_n; z) = (-1)^n\tilde{M}(a_n, b_n; z)$ in equation (A.3). Instead of equation (A.4) we have

$$\alpha_{-1}(2i\omega x)^{-1}b_0\tilde{M}(\nu + i\eta, 2\nu; -2i\omega x) + (\alpha_0b_1 + \beta_0b_0)\tilde{M}(\nu + 1 + i\eta, 2\nu + 2; -2i\omega x) + (\alpha_1b_2 + \beta_1b_1 + \gamma_1b_0)(2i\omega x)\tilde{M}(\nu + 2 + i\eta, 2\nu + 4; -2i\omega x) + \sum_{n=2}^{\infty}(\alpha_nb_{n+1} + \beta_nb_n + \gamma_nb_{n-1})(2i\omega x)^n \times\tilde{M}(n + \nu + 1 + i\eta, 2n + 2\nu + 2; -2i\omega x) = 0.$$

The results are the same as in the previous case, the only technical difference being that to find the recurrence relations for $B_2 = 1$ and $B_2 = 2$ we must use [16]

$$\lim_{b\to 1-n}\frac{M(a, b; z)}{\Gamma(b)} = \frac{\Gamma(a + n)}{\Gamma(a)\Gamma(n + 1)}z^nM(a + n, 1 + n; z)$$

which yields

$$\lim_{b\to -1}\tilde{M}(i\eta - 1/2, b; -2i\omega x) = (2i\omega x)^2\tilde{M}(i\eta + 3/2, 3; -2i\omega x),$$

$$\lim_{b\to 0}\tilde{M}(i\eta, b; -2i\omega x) = (2i\omega x)\tilde{M}(1 + i\eta, 2; -2i\omega x).$$

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